Indirect Proofs

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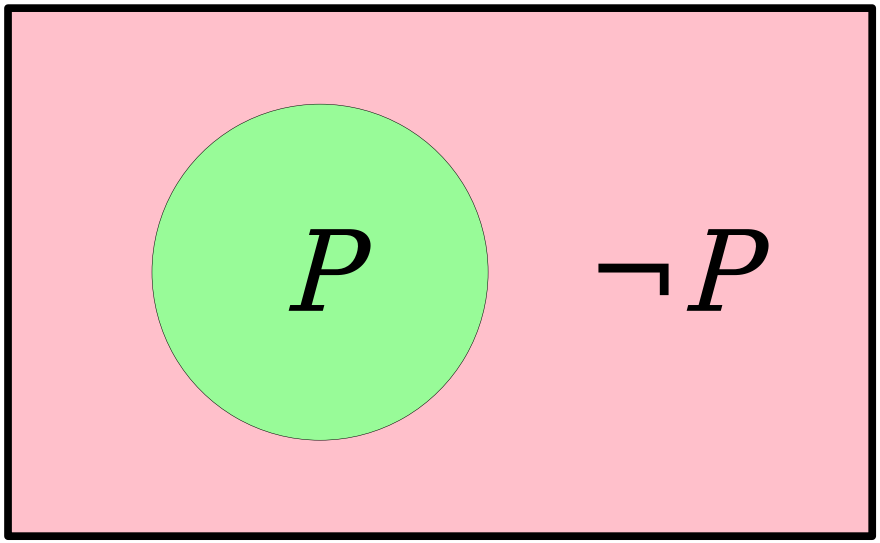
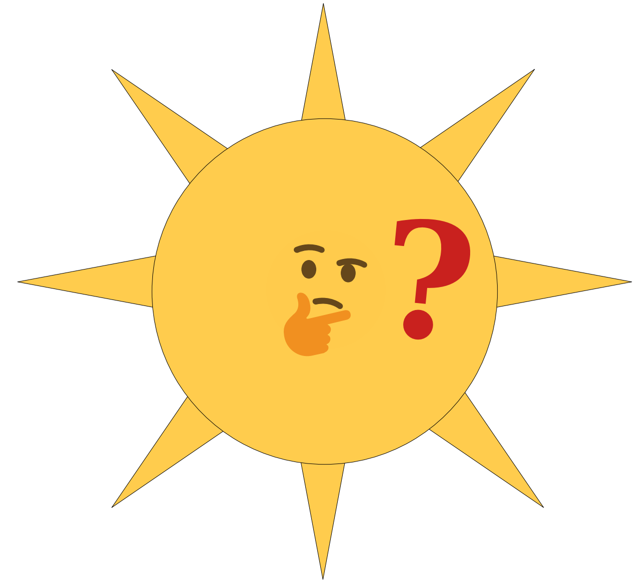
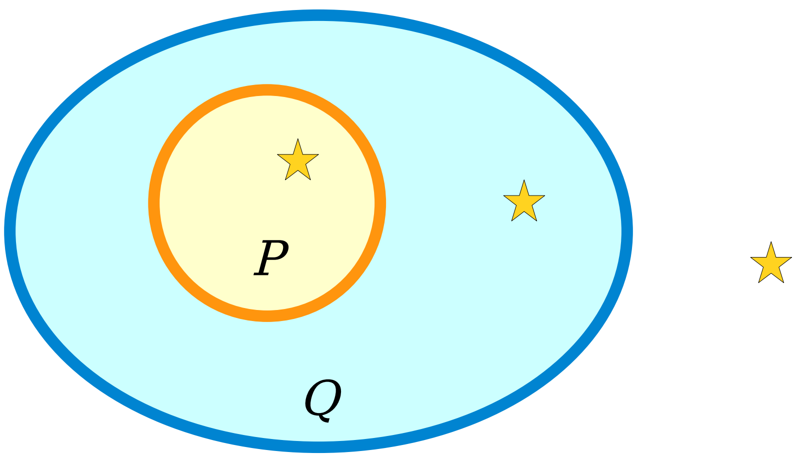
A Story in Four Acts



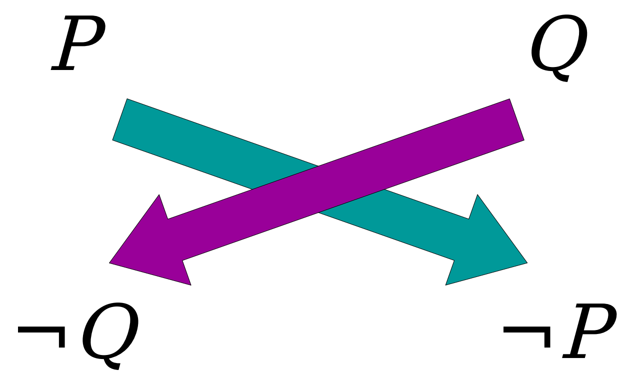
Logical Negation



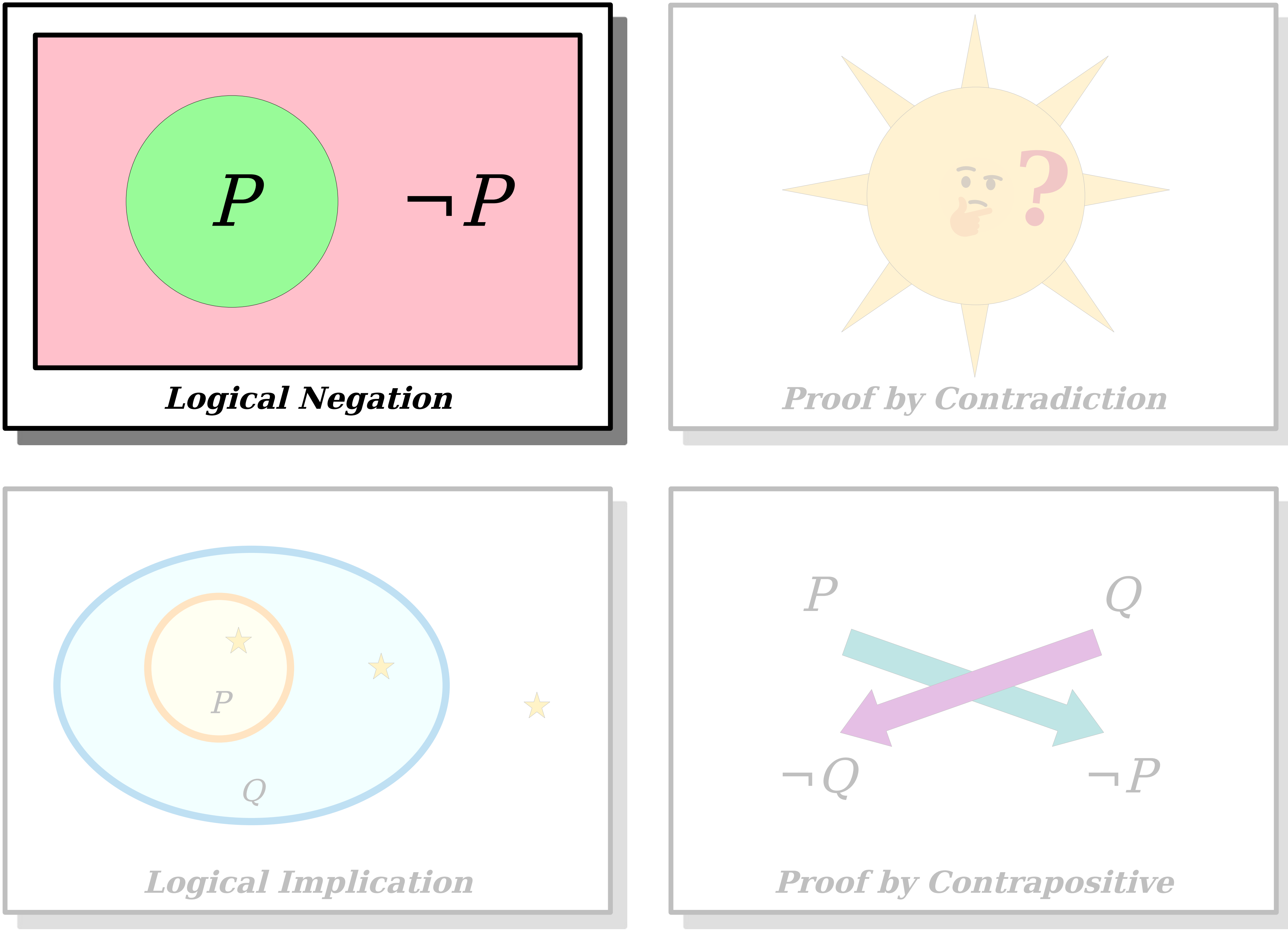
Proof by Contradiction



Logical Implication



Proof by Contrapositive


P
¬P
Logical Negation
?
Proof by Contradiction
P
Q
Logical Implication
P
Q
¬Q
¬P
Proof by Contrapositive

Act I

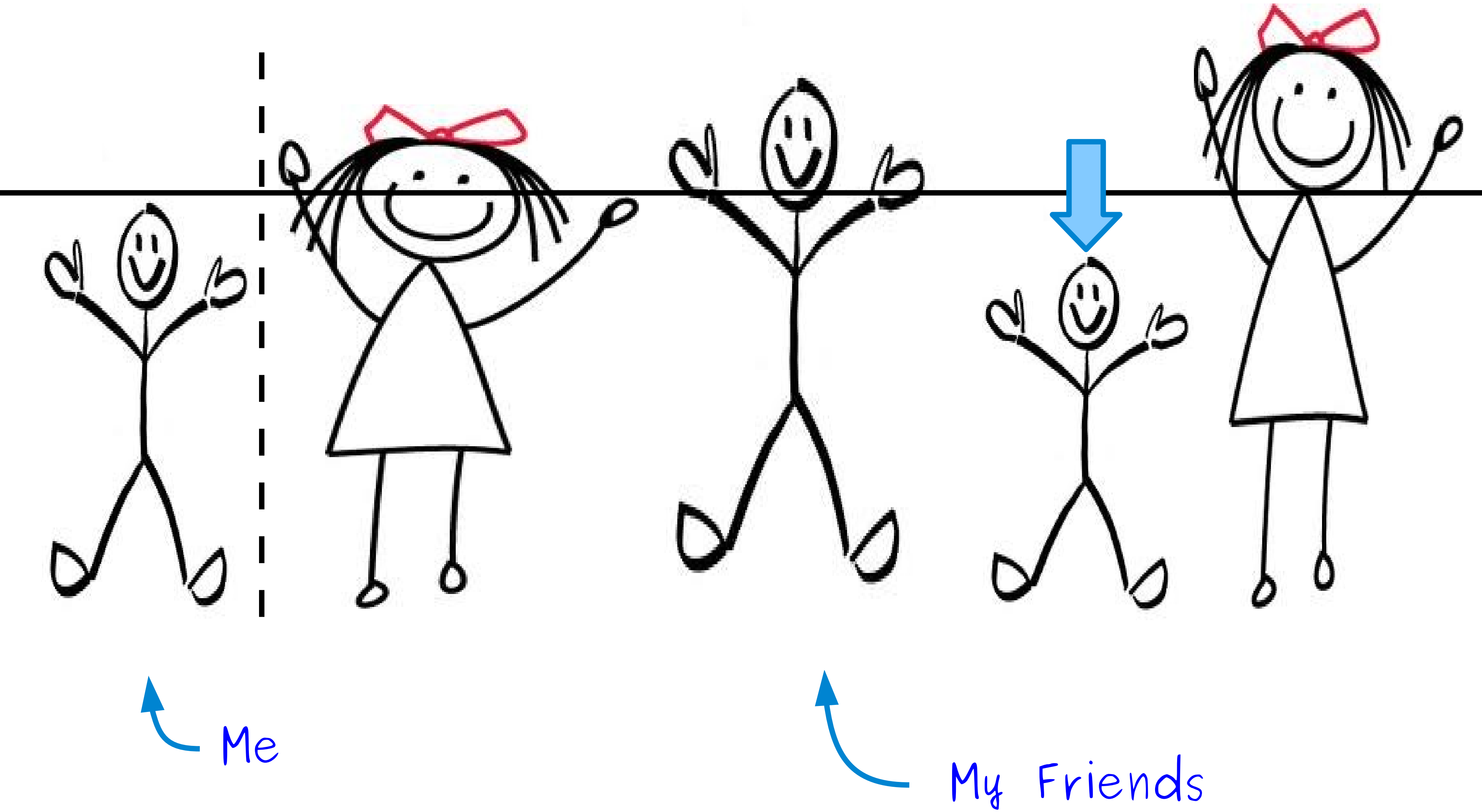
Logical Negation

Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
- The ***negation*** of a proposition X is a proposition that is true when X is false and is false when X is true.
- For example, consider the proposition "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside."
  - Its negation is *not* "we're in the Bay Area."

How do you find the negation of a statement?

"All My Friends Are Taller Than Me"



The negation of the ***universal*** statement

Every *P* is a *Q*

is the ***existential*** statement

There is a *P* that is not a *Q*.

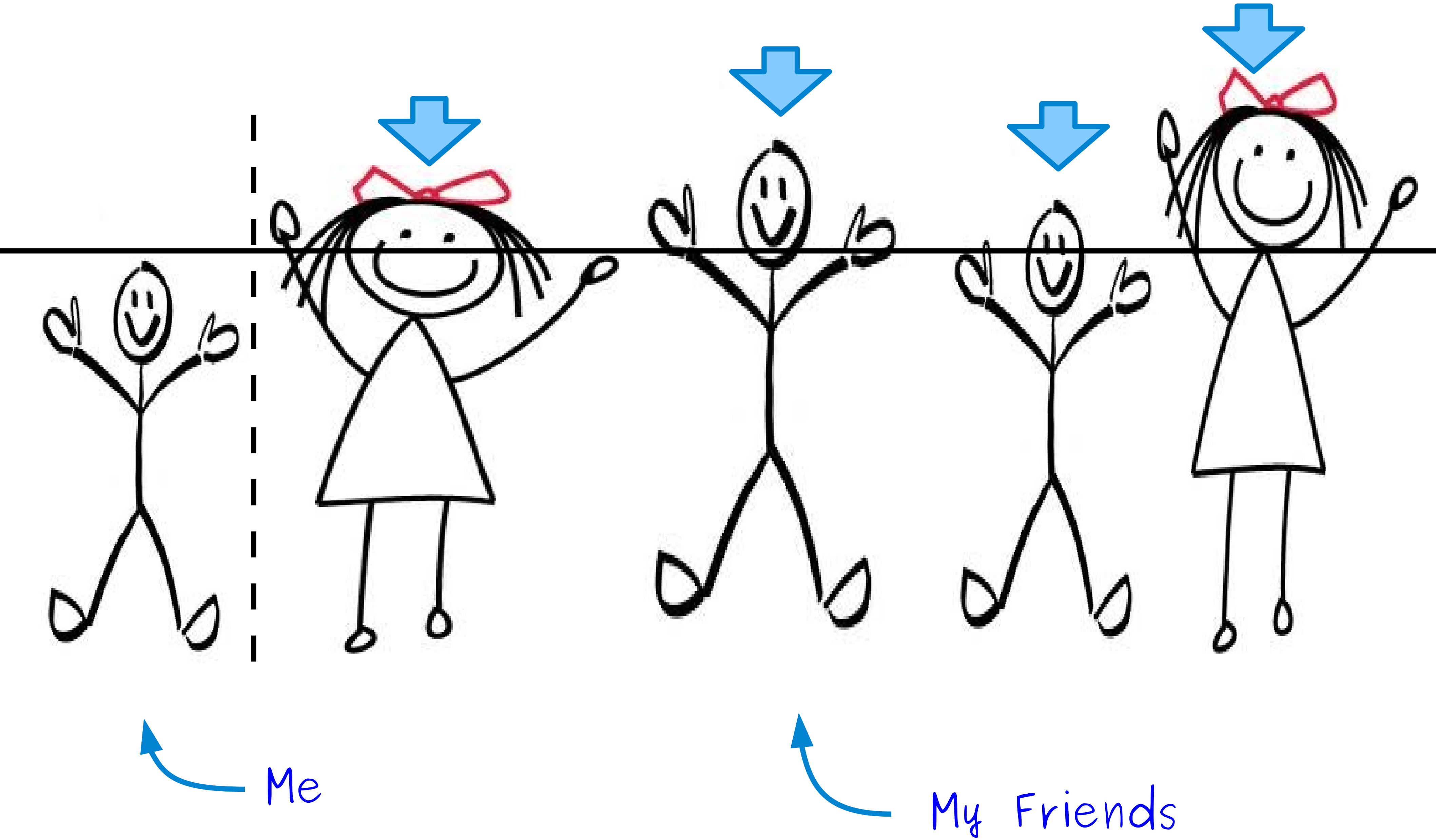
The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the ***existential*** statement

There exists an *x* where *P*(*x*) is false.

“Some Friend Is Shorter Than Me”



The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the ***universal*** statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

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For all *x*, *P*(*x*) is false.

Your Turn!

- What's the negation of the following statement?

"Every brown dog loves every orange cat."

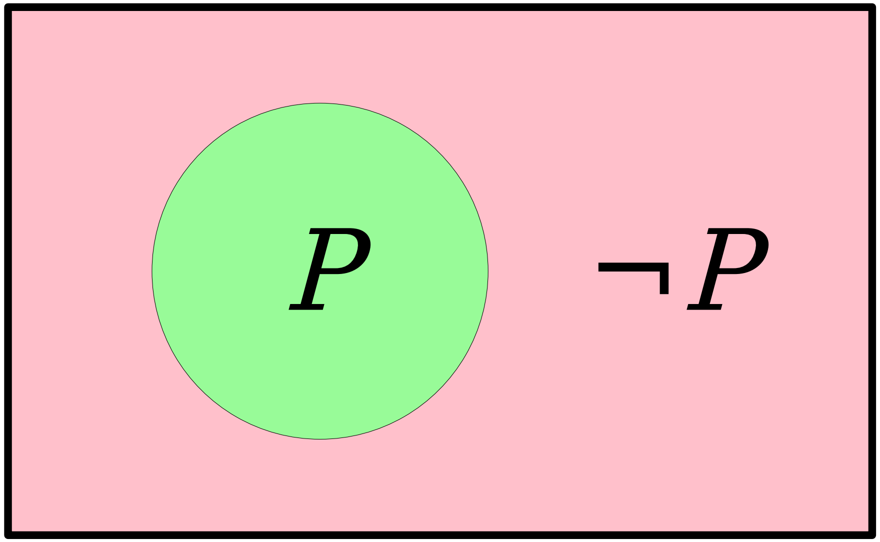
Your Turn!

- What’s the negation of the following statement?

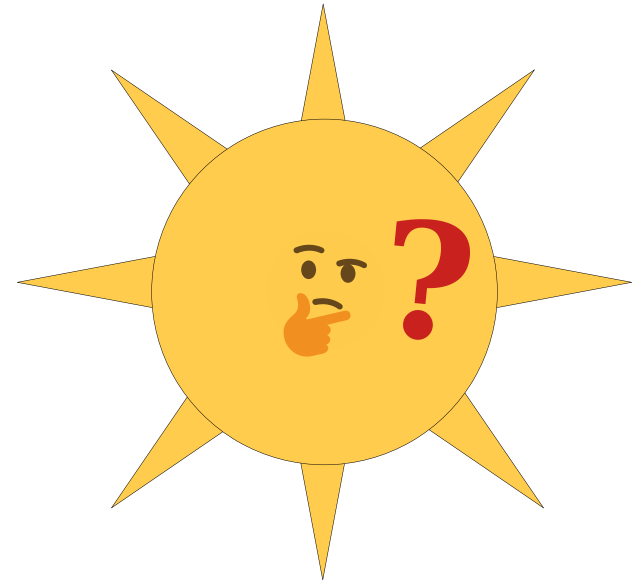
“Every brown dog loves every orange cat.”

- Answer:

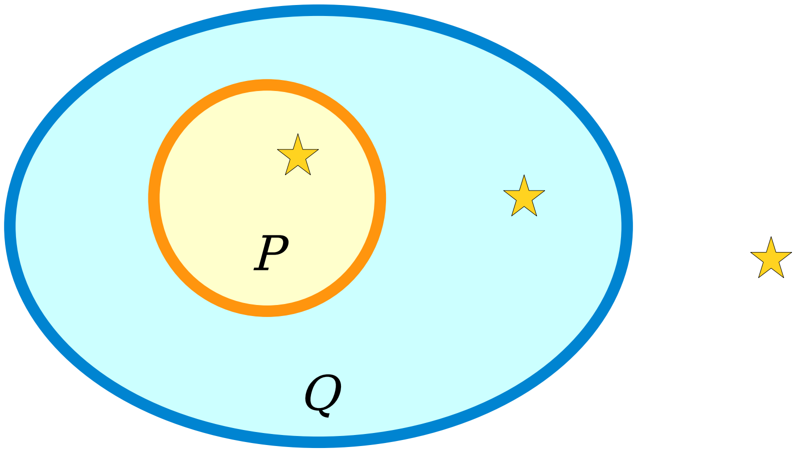
“There is a brown dog that doesn’t love some orange cat”



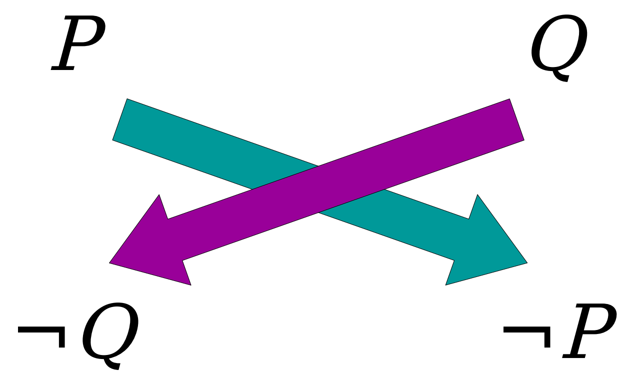
Logical Negation



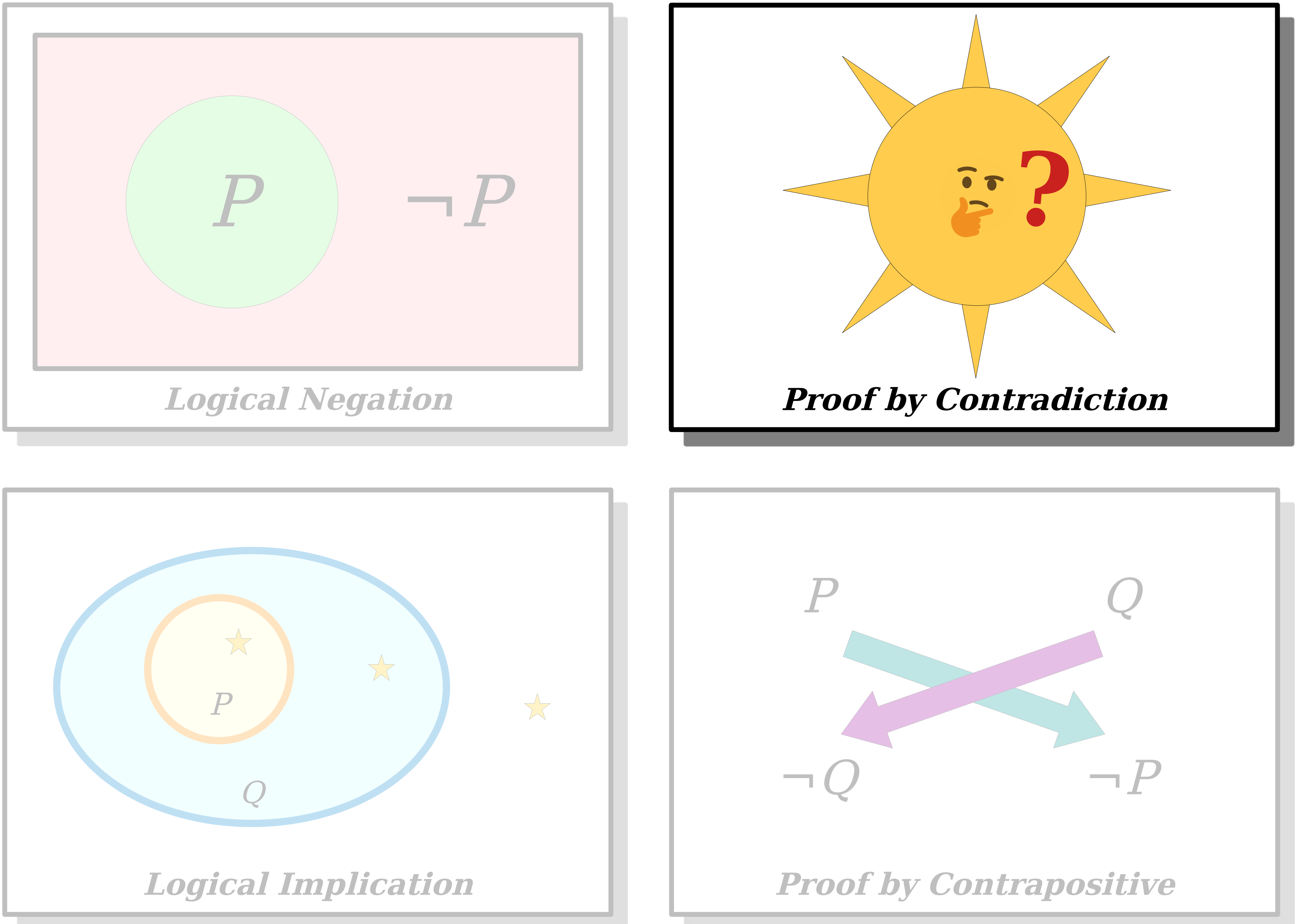
Proof by Contradiction



Logical Implication



Proof by Contrapositive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Act II

Proof by Contradiction

First, let's reflect on the **direct proof** technique we saw Wednesday.

Our First Proof! (from Wednesday)

Theorem: If n is an even integer, then n^2 is even.

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Since n is even, there is some integer k such that $n = 2k$.

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$$n^2 = (2k)^2$$

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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \end{aligned}$$

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Since n is even, there is some integer k such that $n = 2k$. This means that

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n

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To prove

"If *P* is true, then *Q* is true,"

we start by asking our reader to assume ***P*** is true.

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If we apply **sound logic** (using definitions, algebra, etc.) all the statements that follow are **also true.**

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From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

More generally speaking,

the process looks like this:

Direct Proof

We start with a statement (or statements) we know (or assume) to be true.

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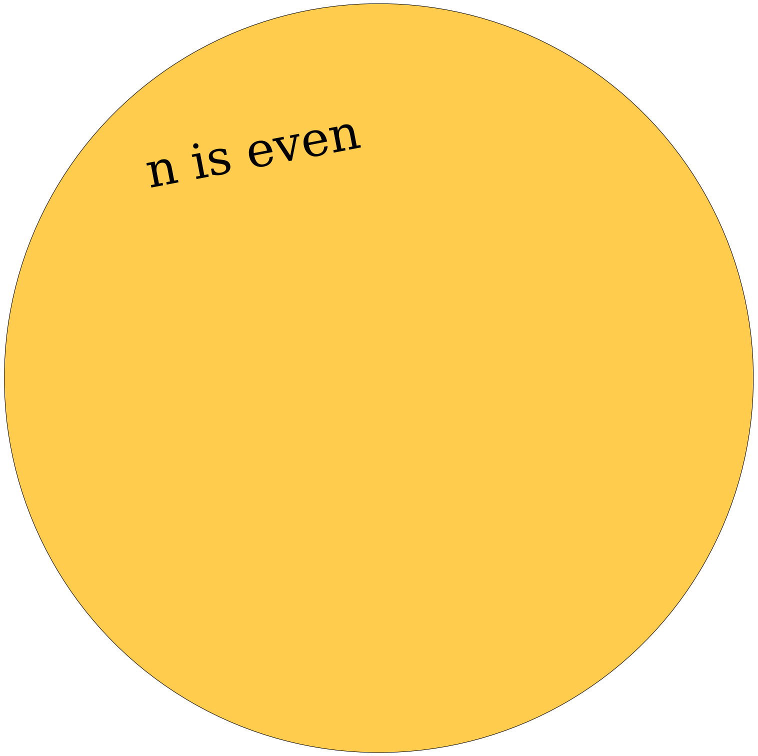


Direct Proof

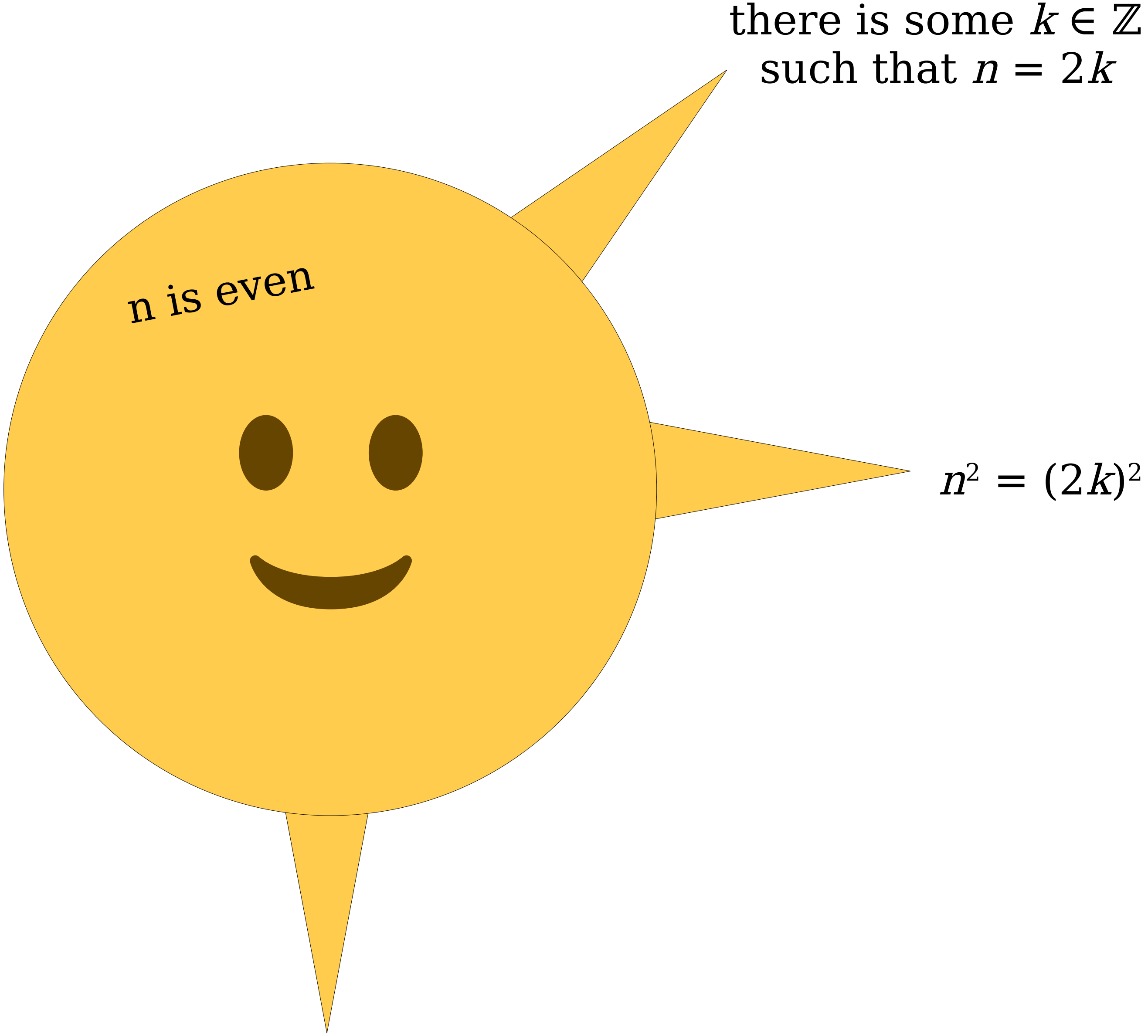
We start with a statement (or statements) we know (or assume) to be true.

Direct Proof

Next, we apply sound logic and rational argument to arrive at other true statements!

Direct Proof

Direct Proof



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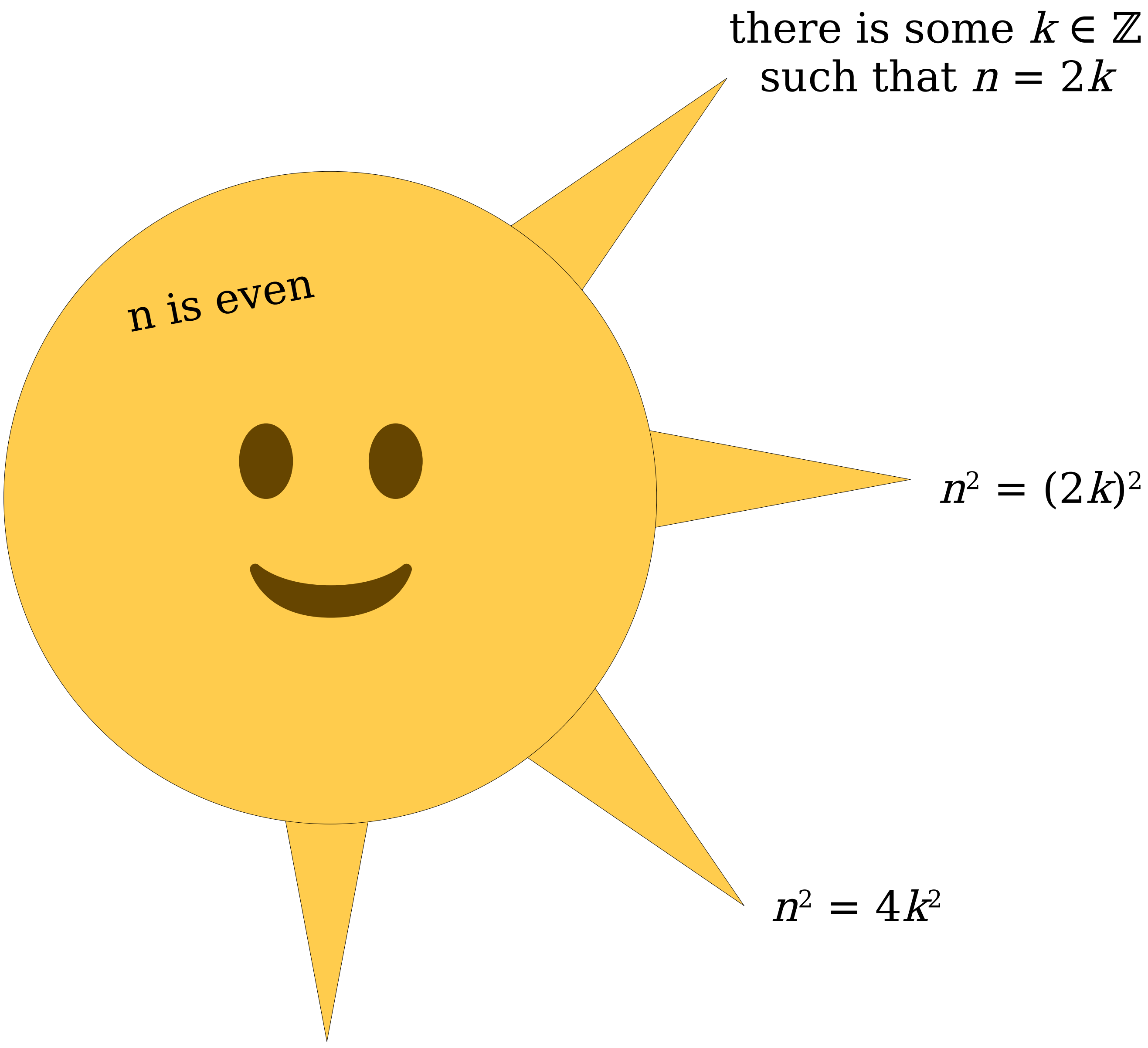

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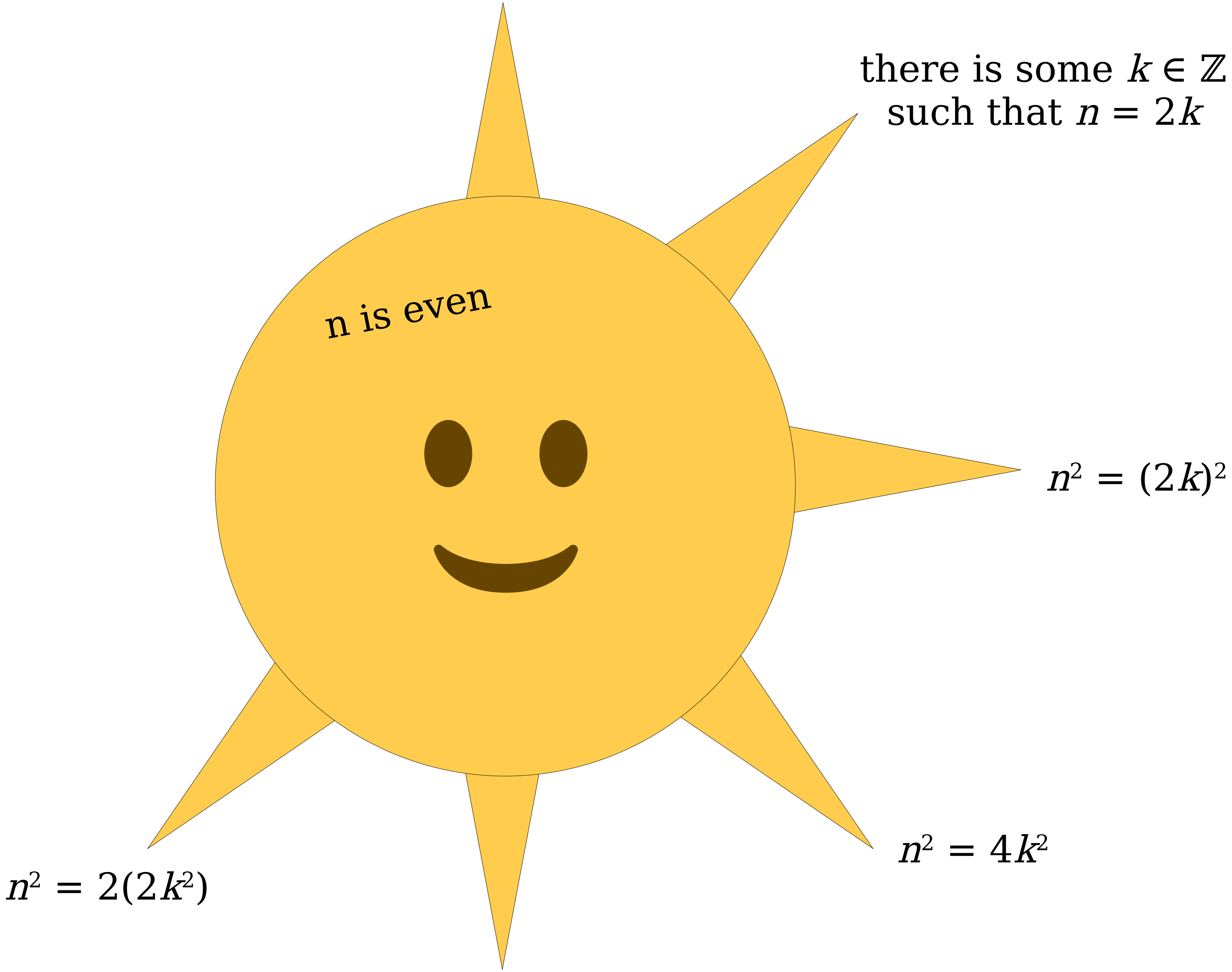

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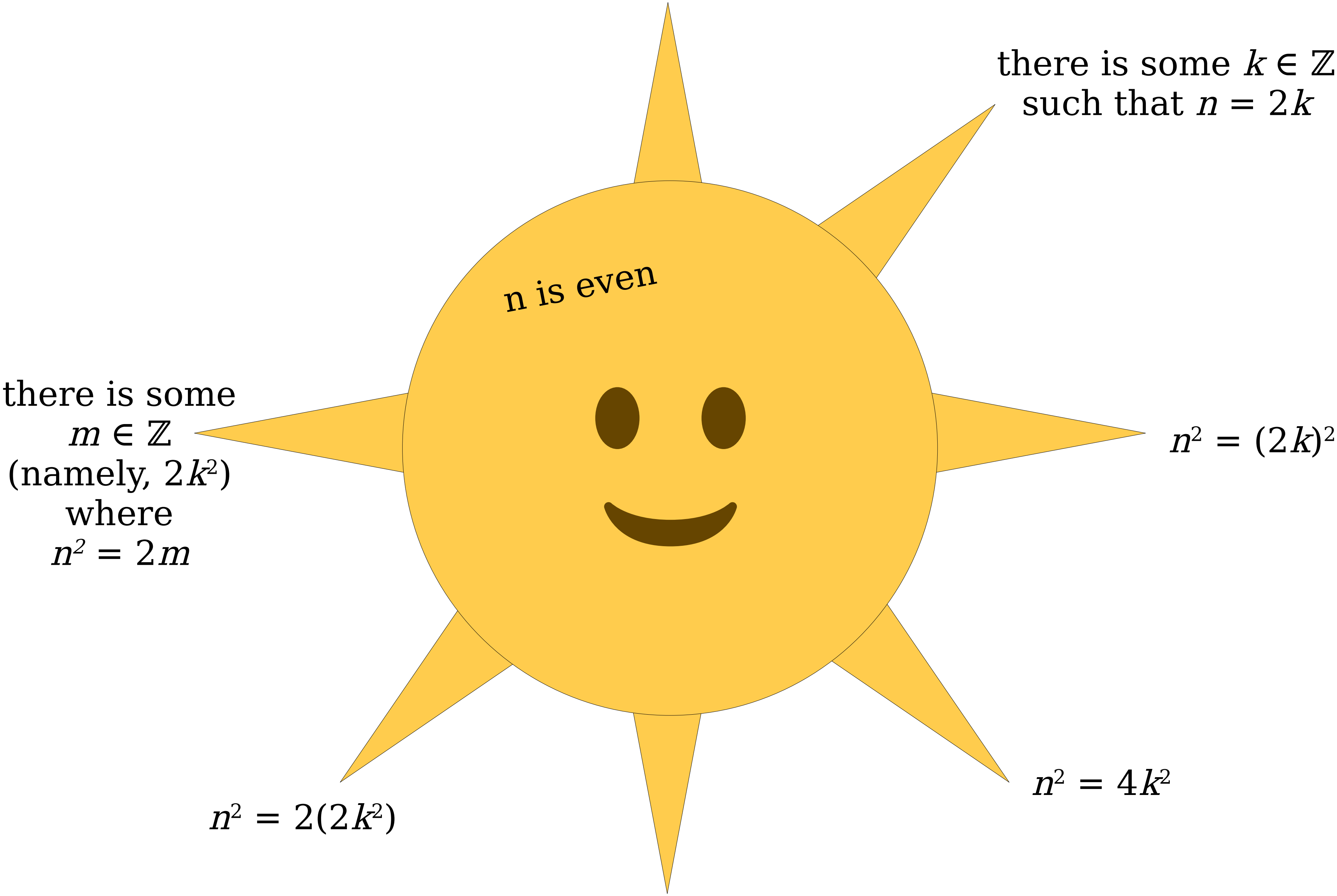

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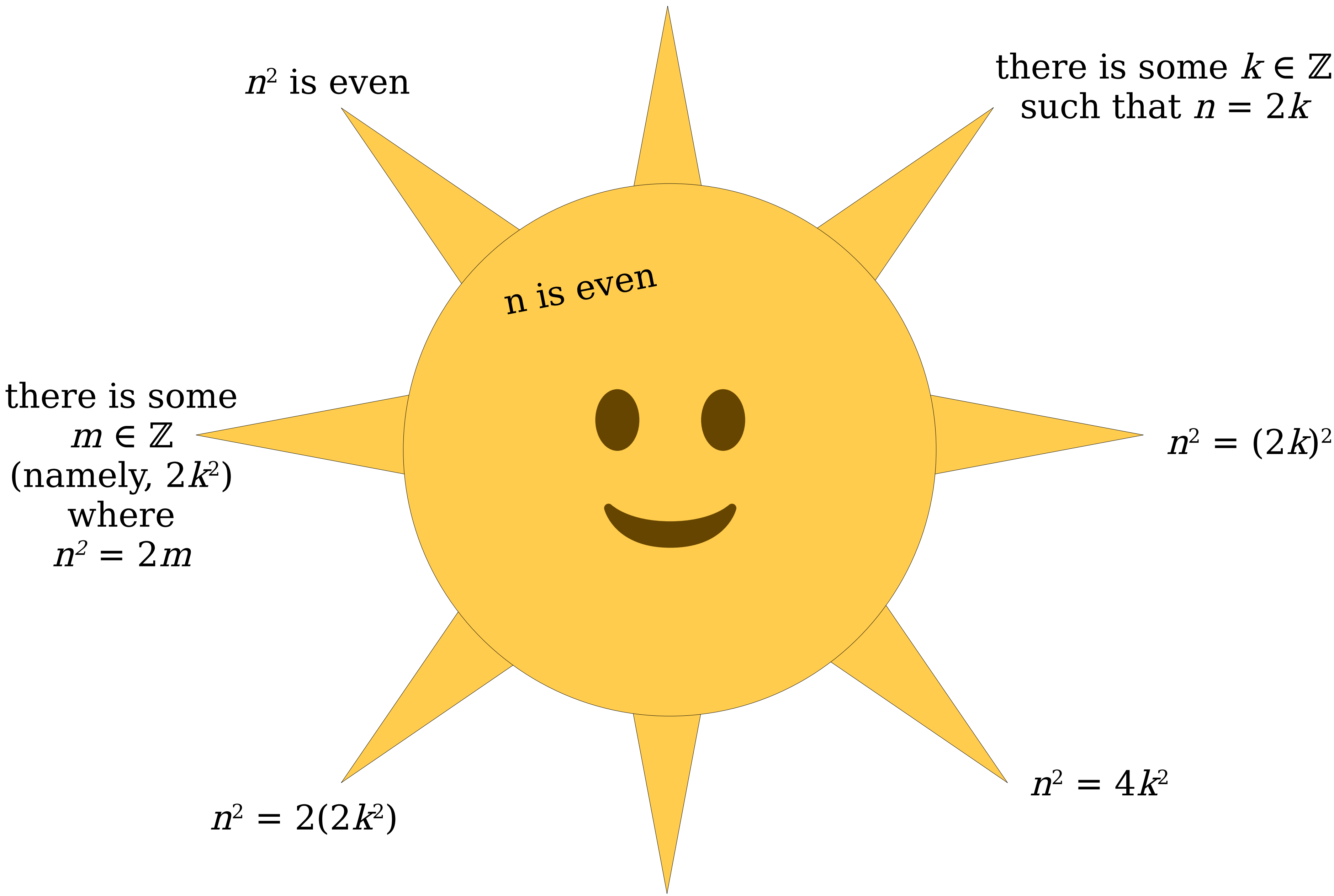

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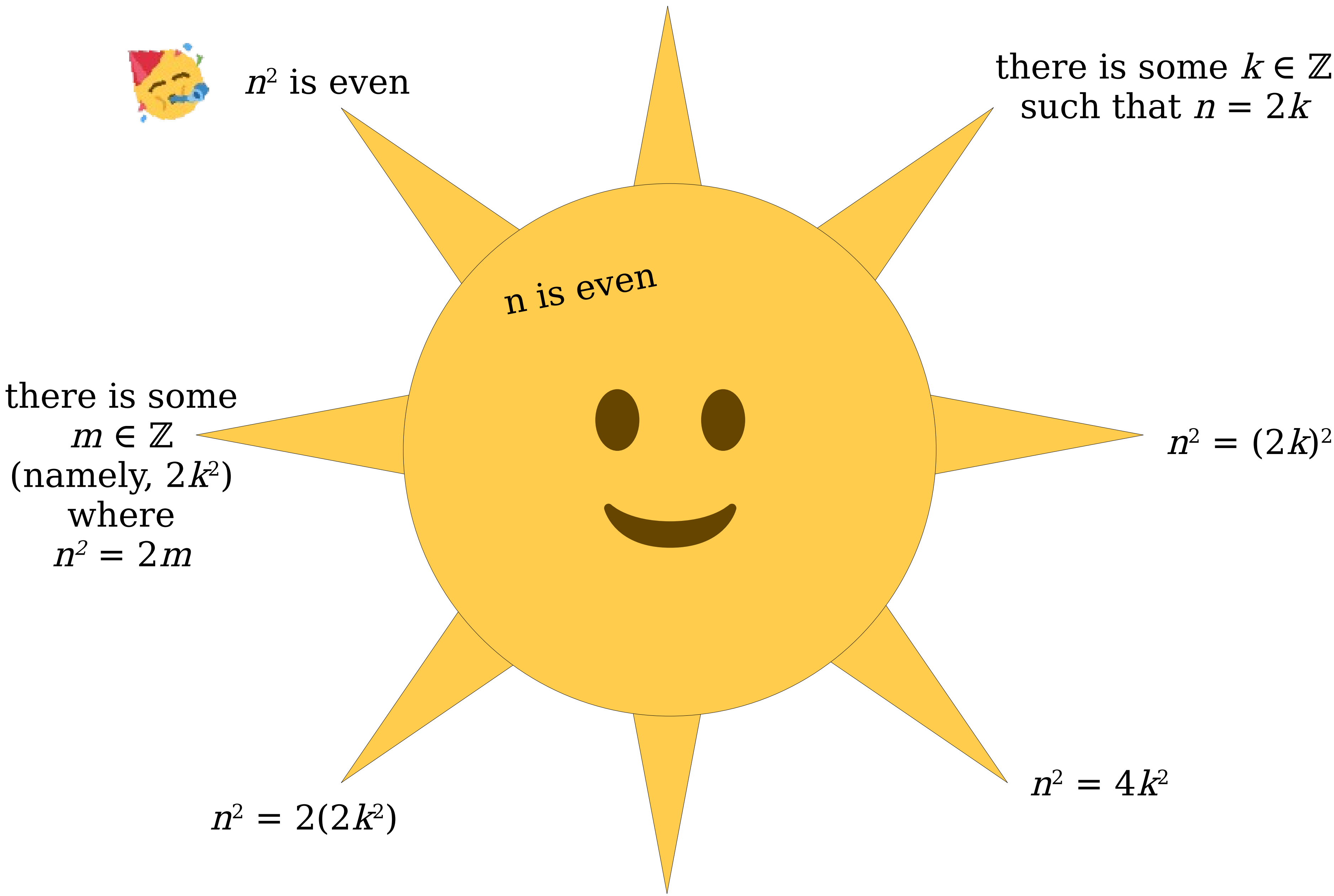

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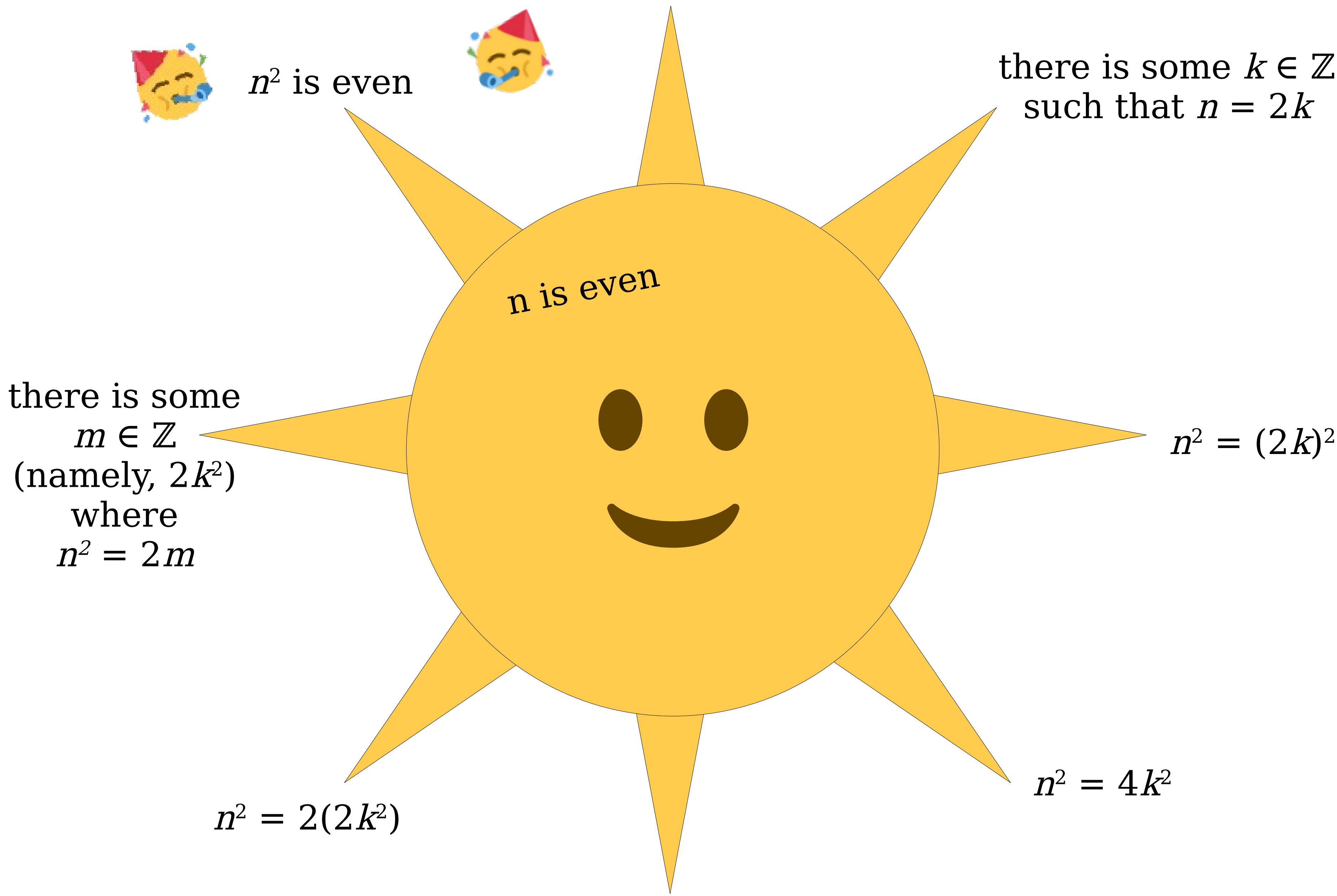

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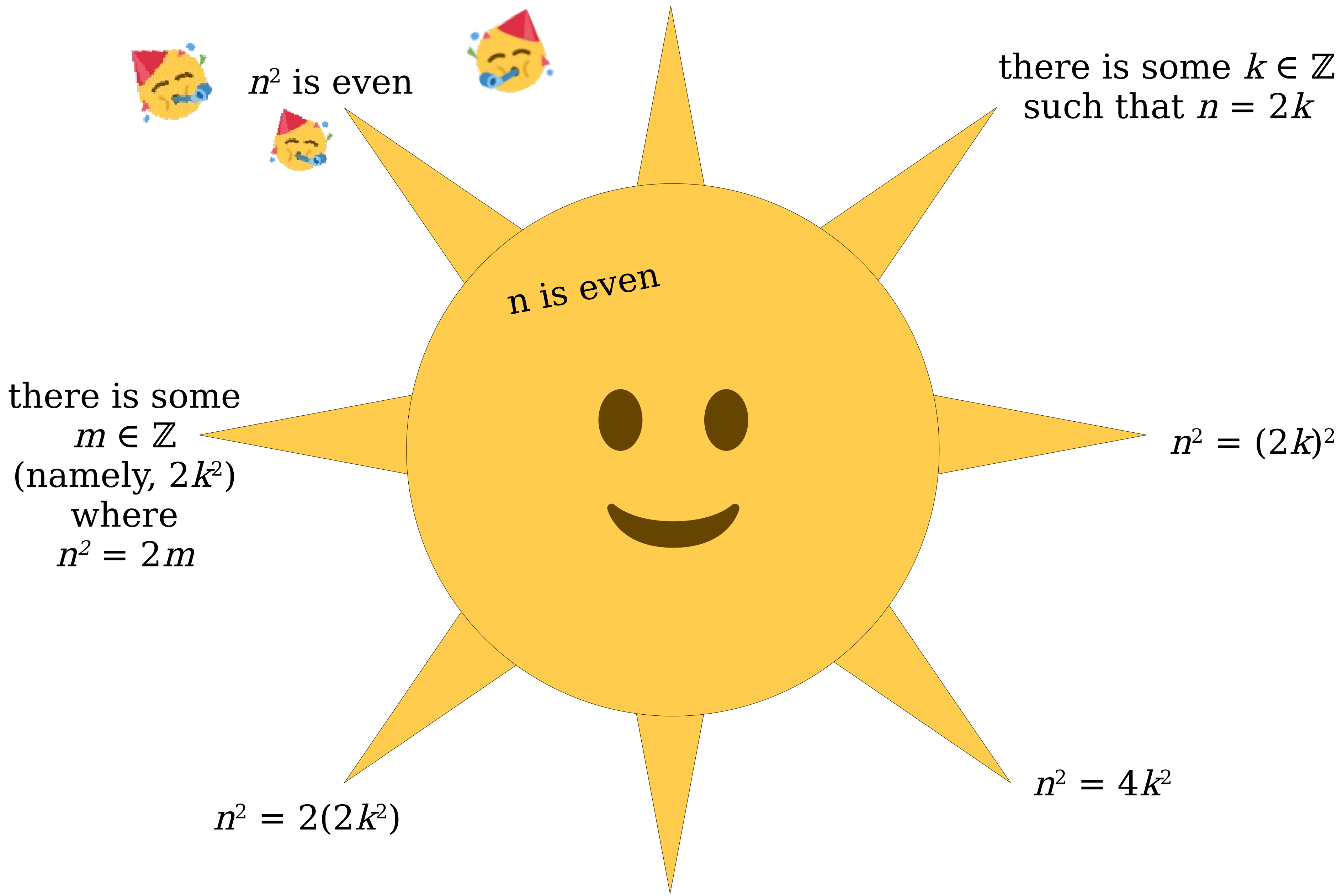

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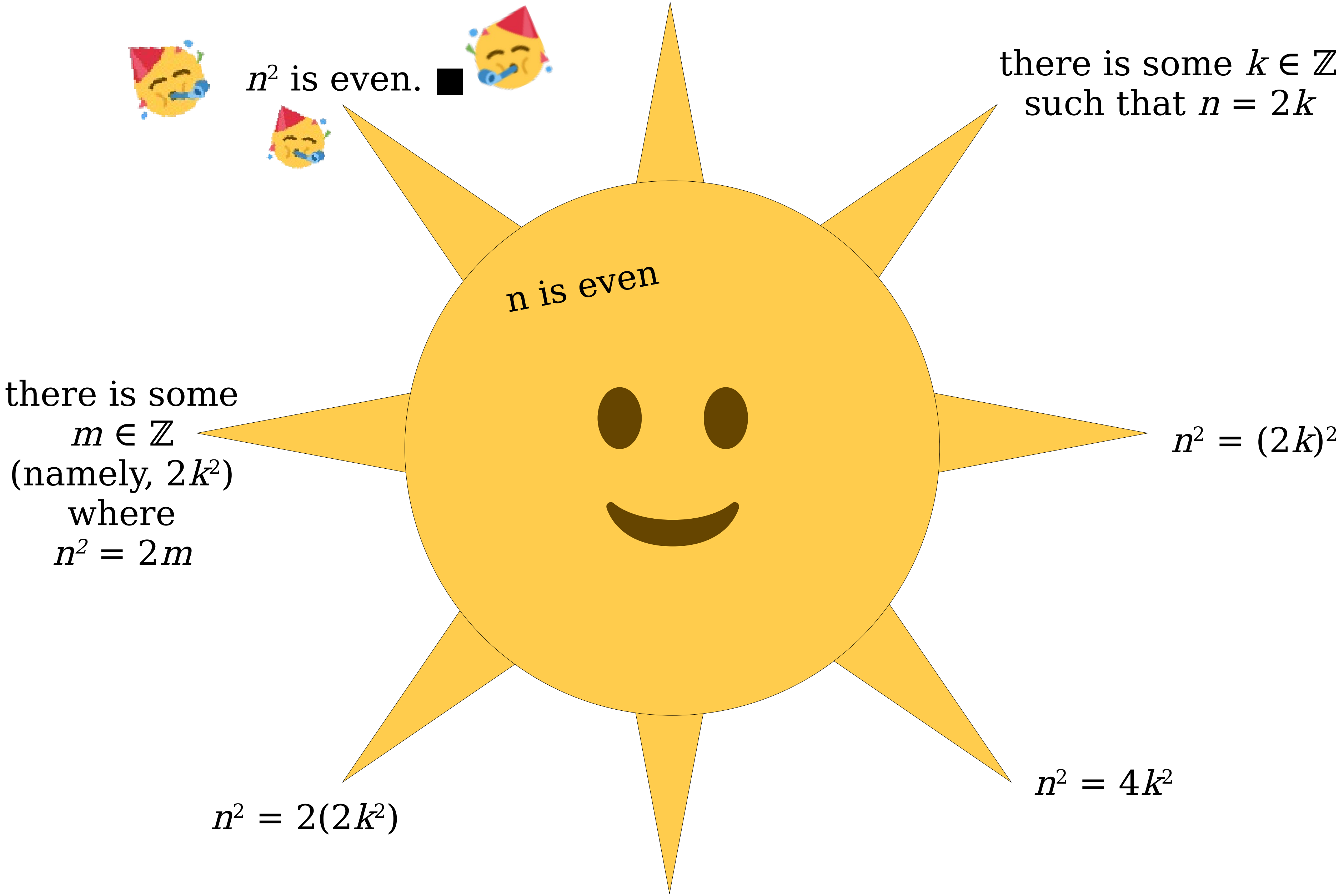

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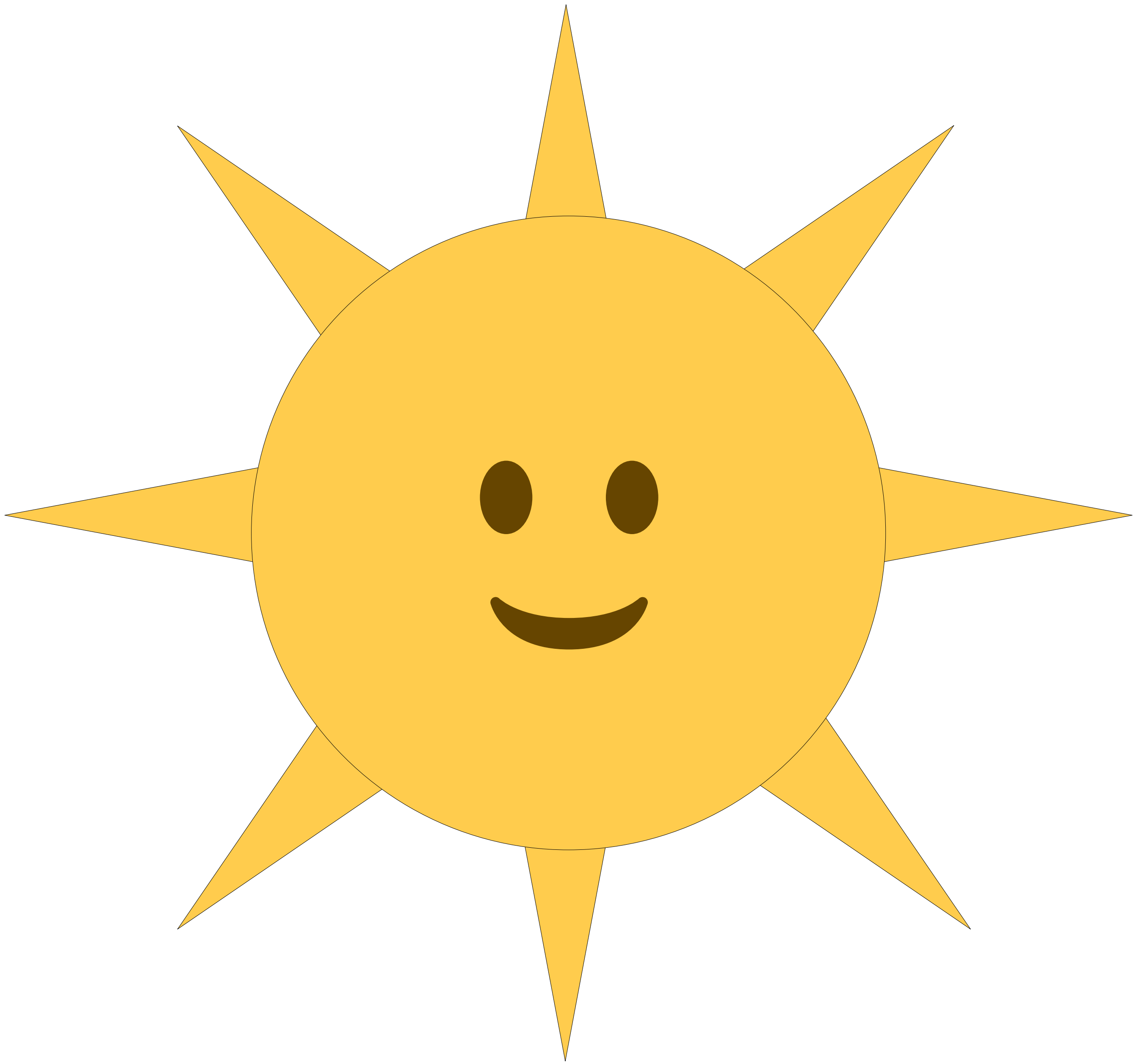

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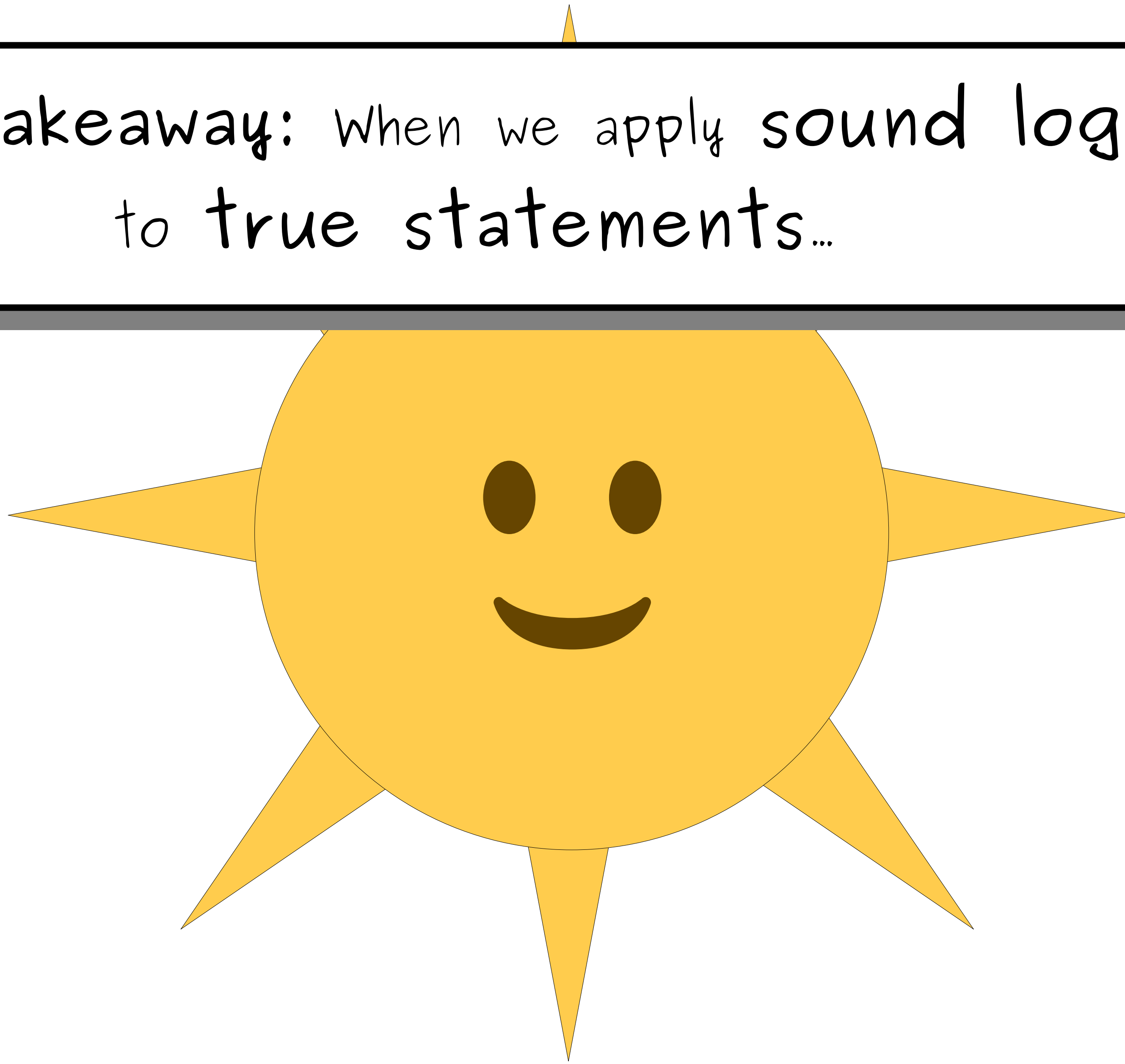
Key Takeaway: When we apply **sound logic** to **true statements**…



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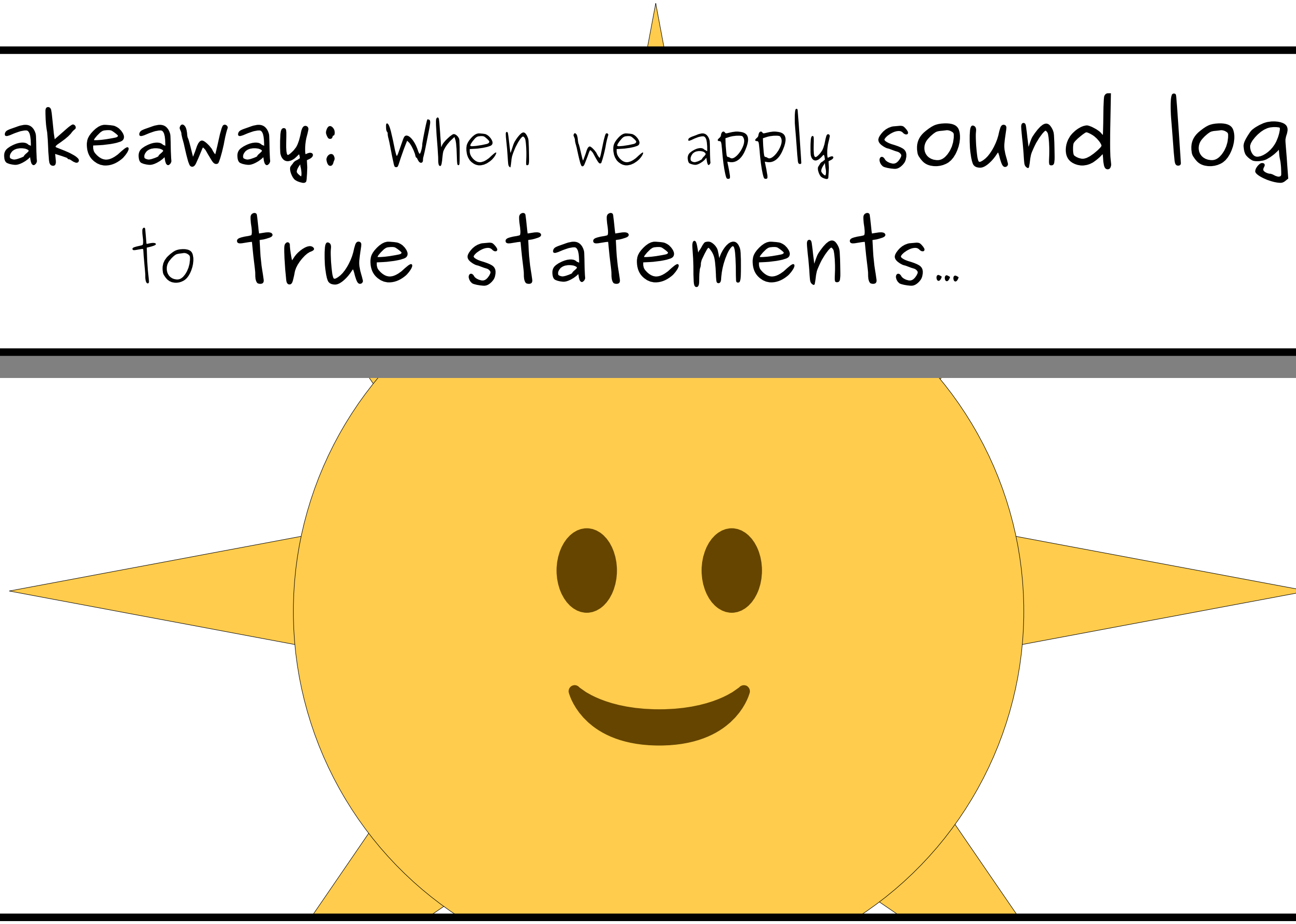

the process **radiates truth** with the power and intensity of a thousand burning suns!

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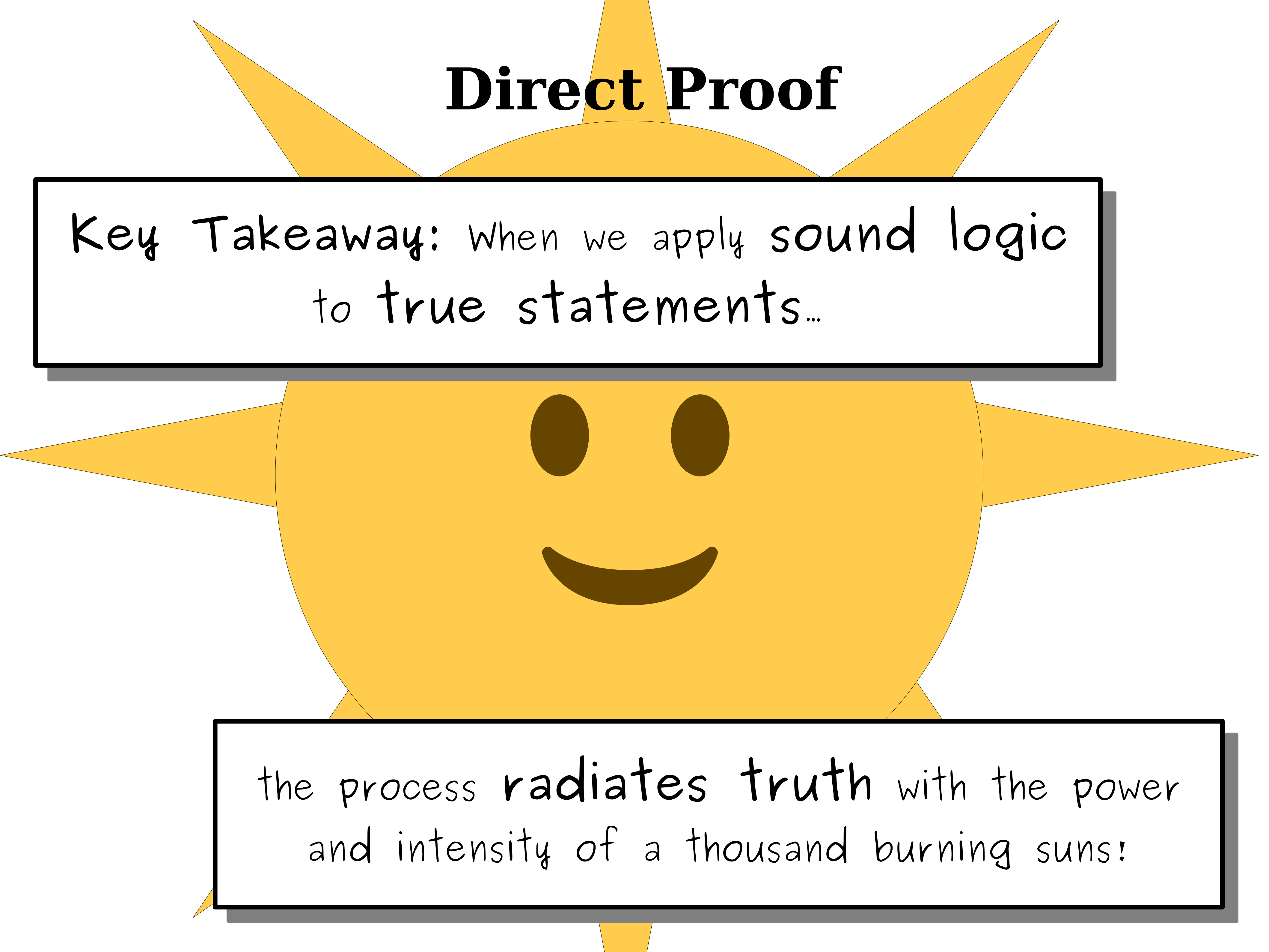

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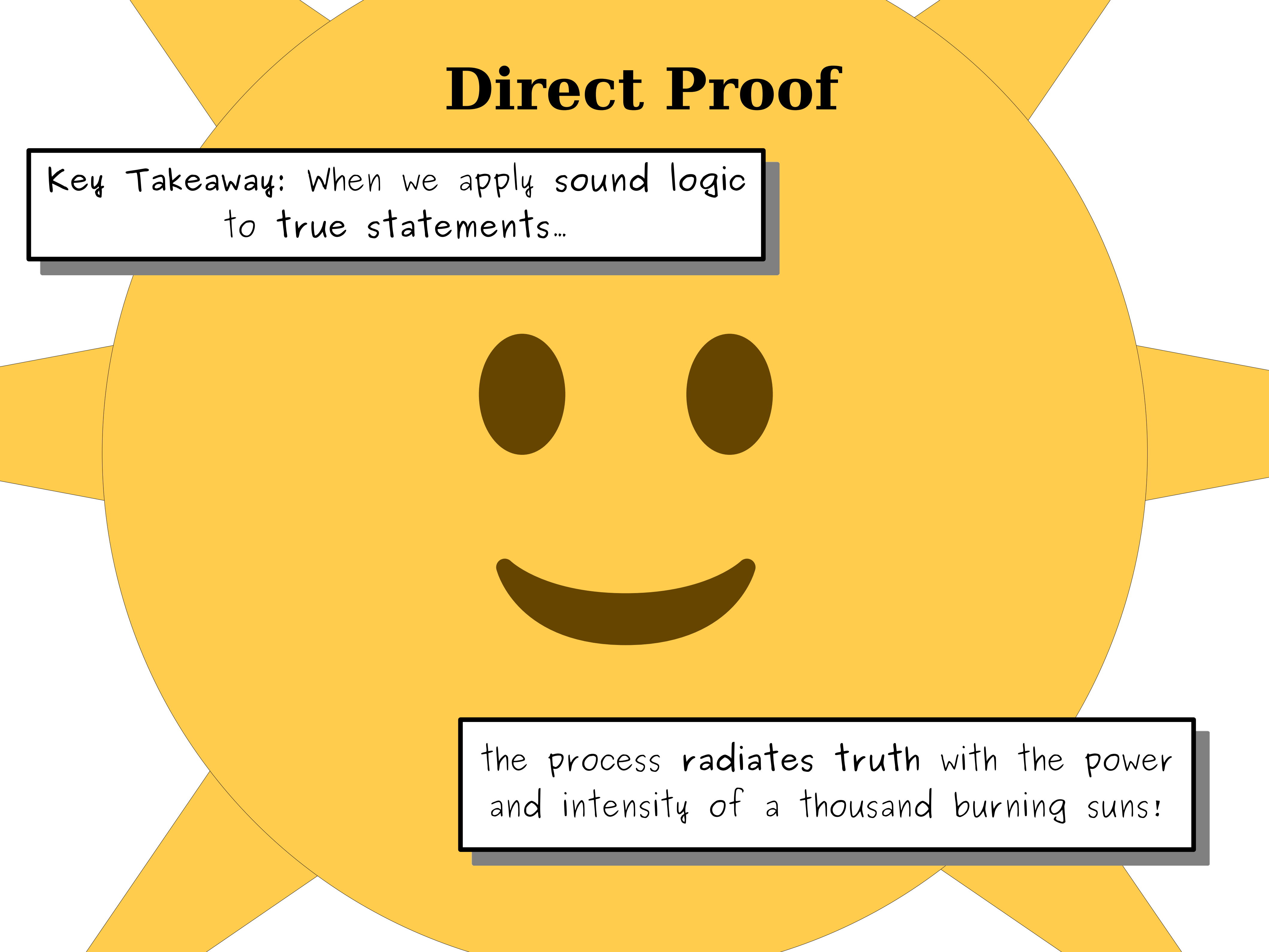
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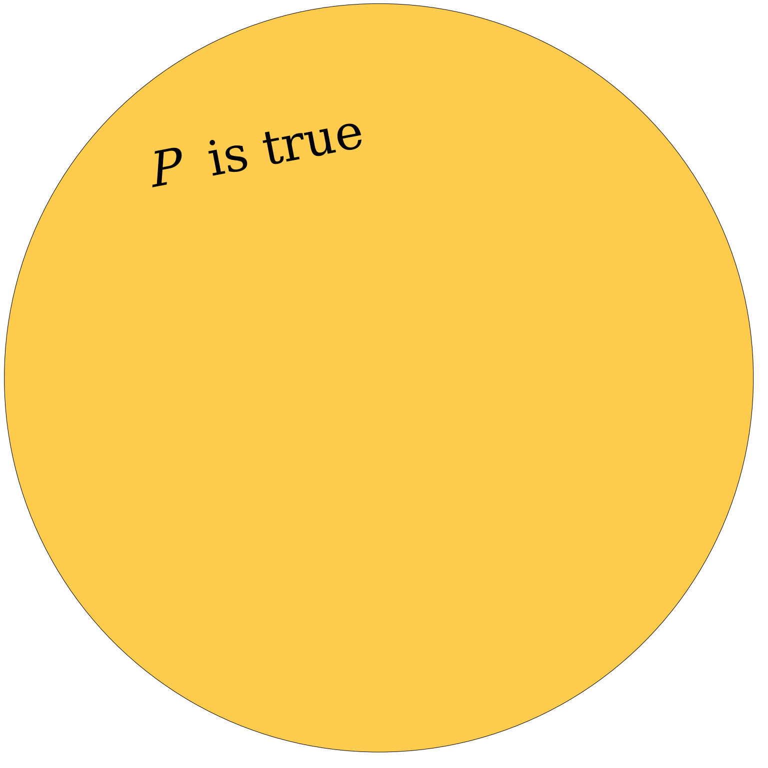
Key Takeaway: When we apply **sound logic** to **true statements**…
the process **radiates truth** with the power and intensity of a thousand burning suns!

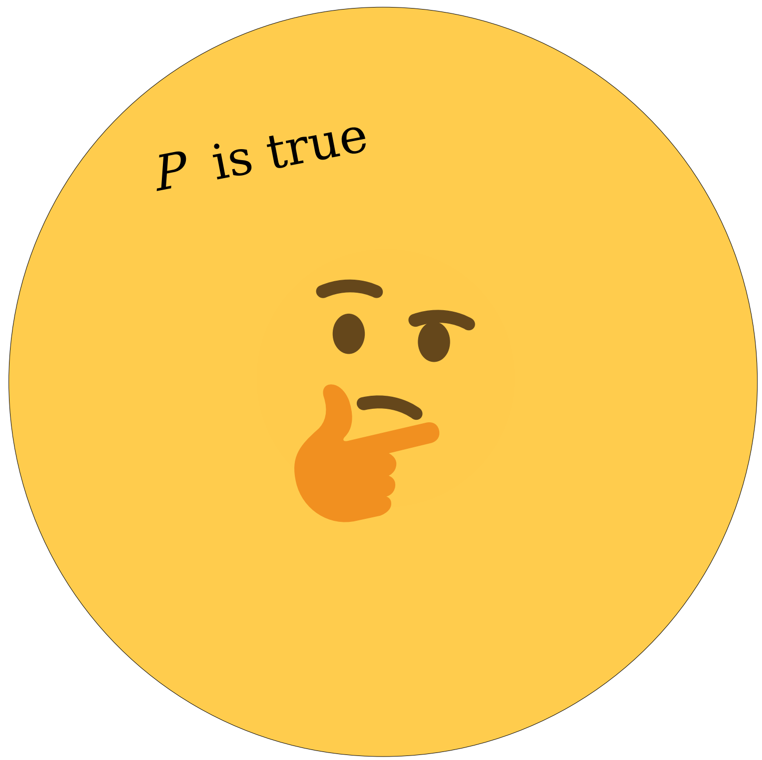
Okay, but…

Okay, but…

what if we start with a proposition

whose **truthiness** is **unknown** to us?


P is true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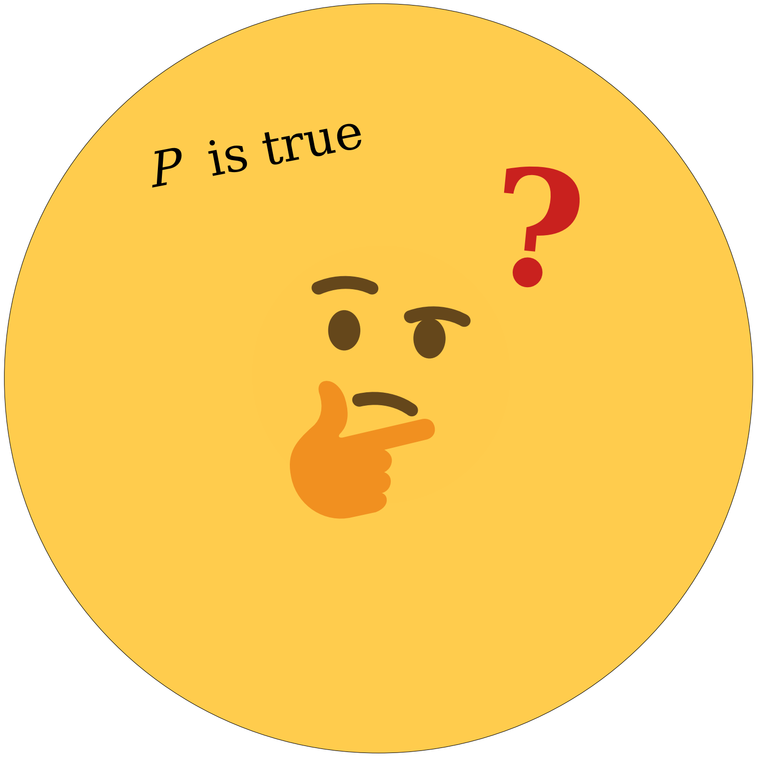
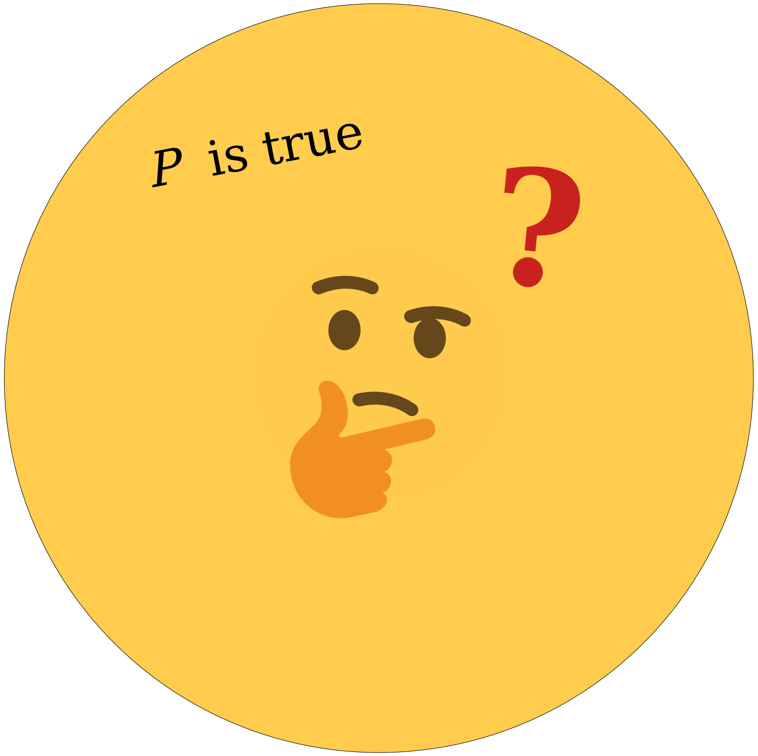


P is true
?

This might radiate SOME truth.



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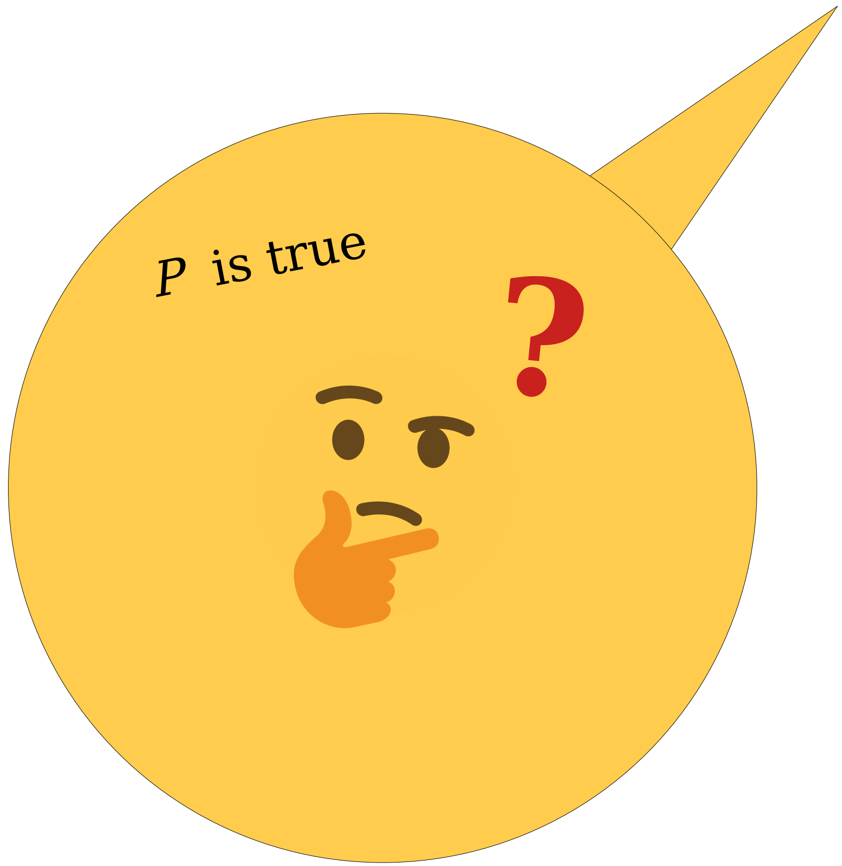

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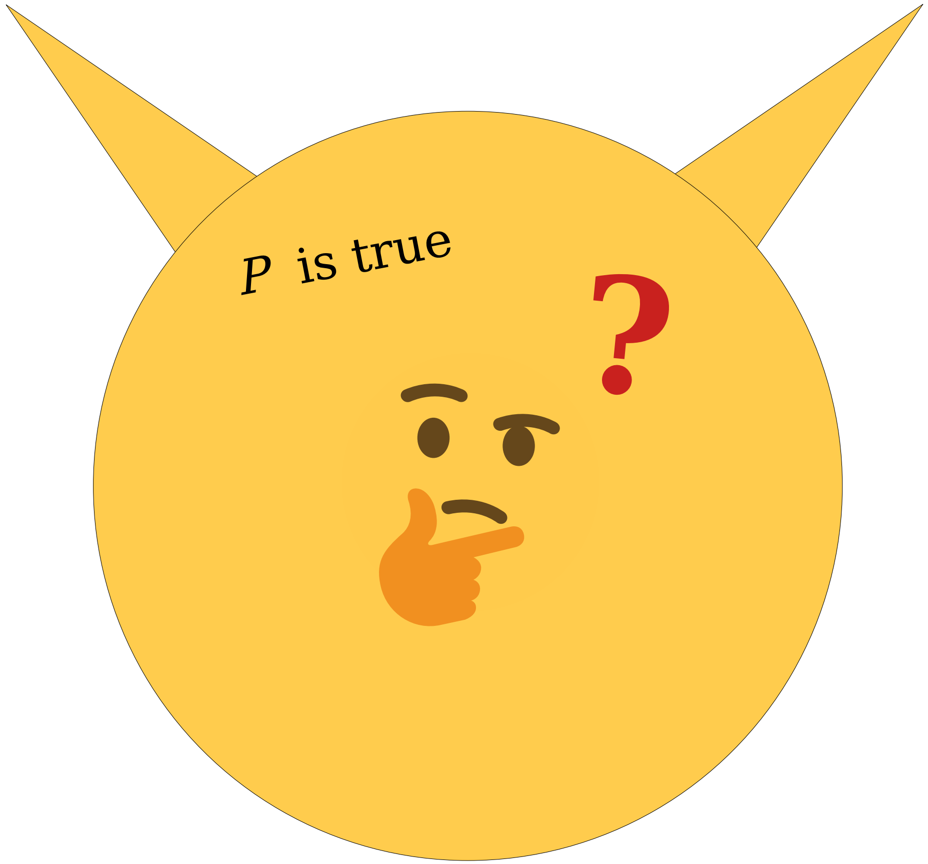

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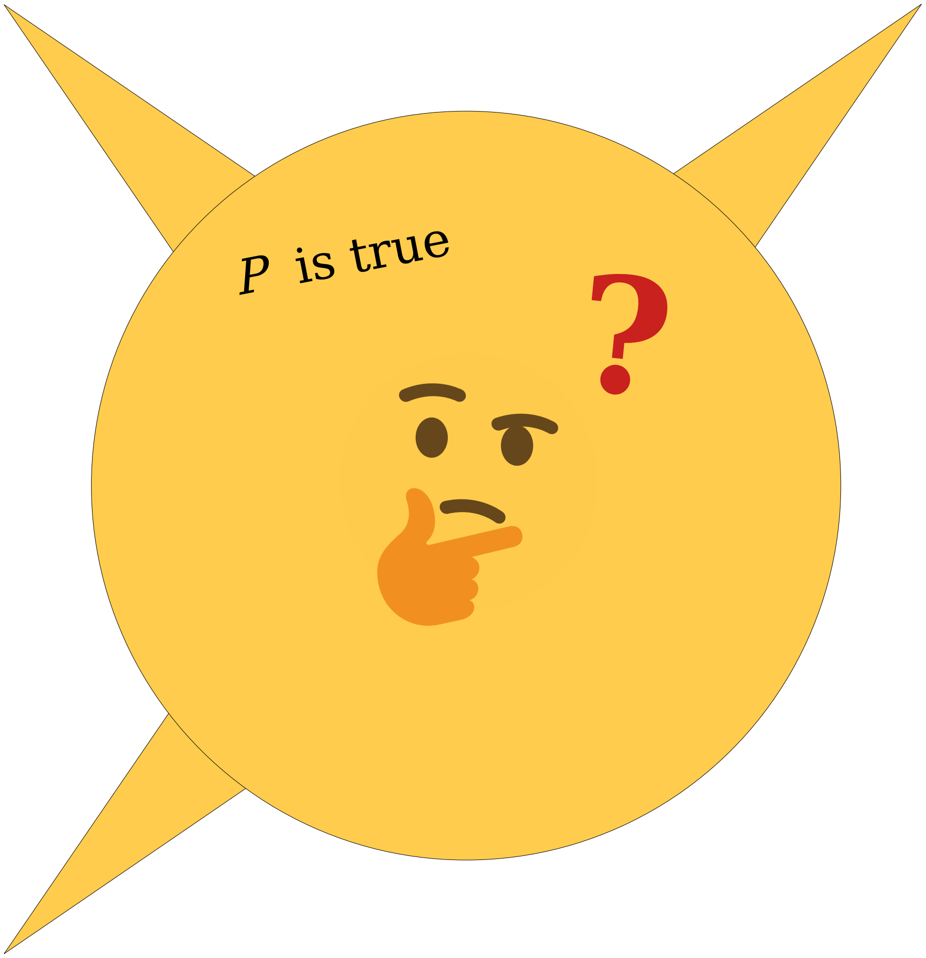

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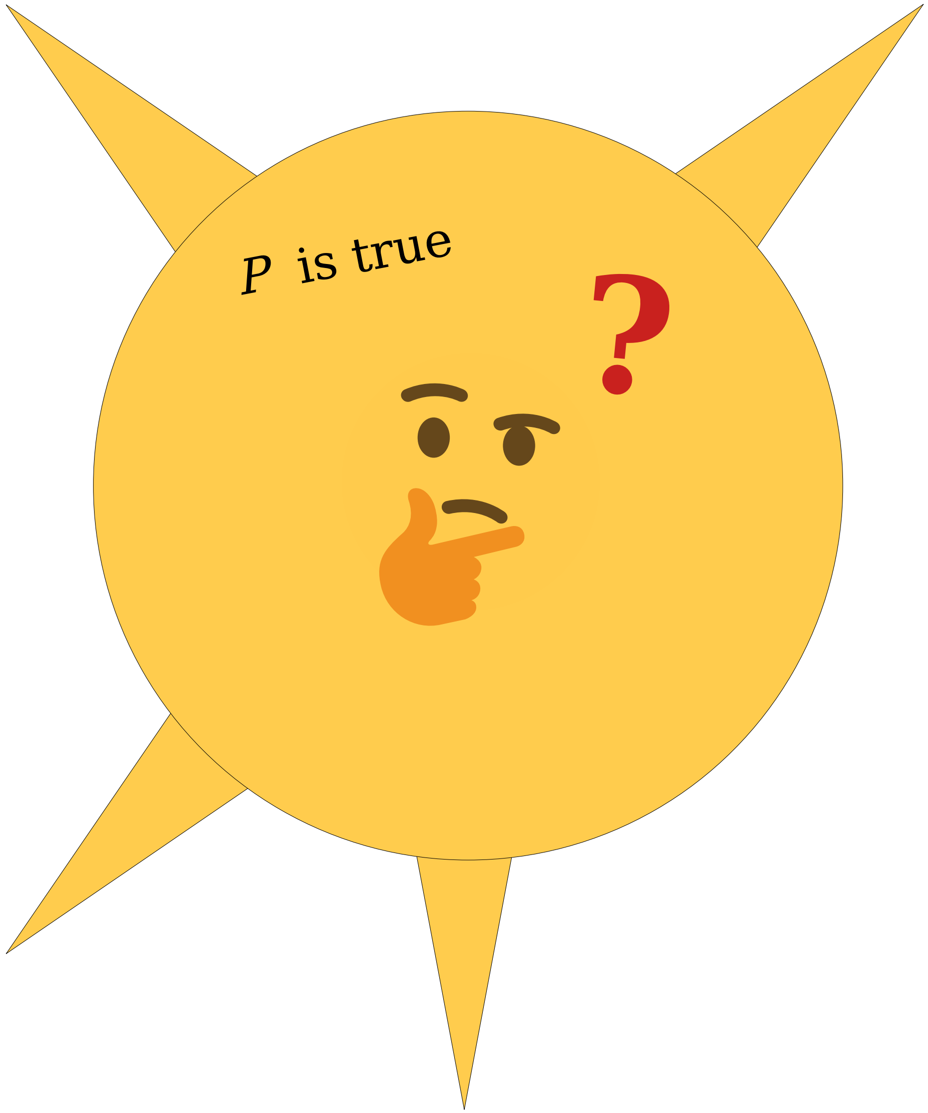

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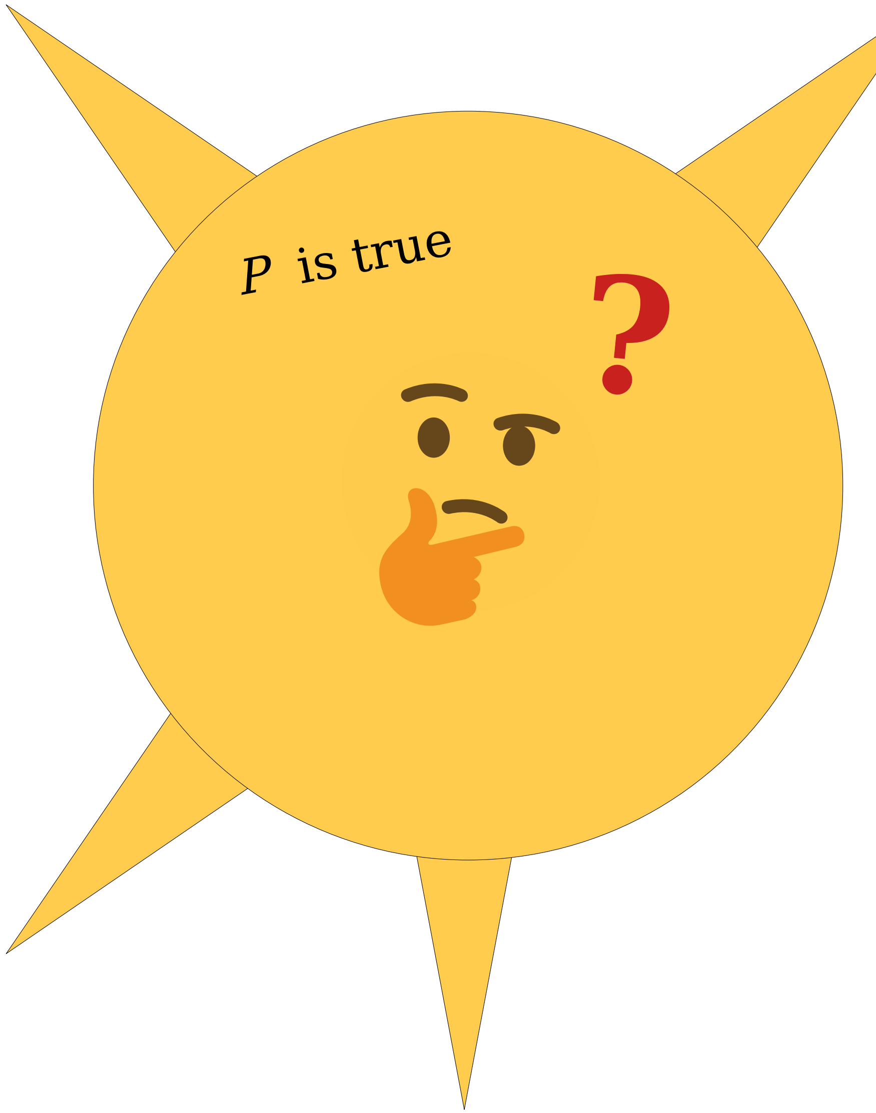
What **does** it mean if we start generating statements we **know** are **false**?



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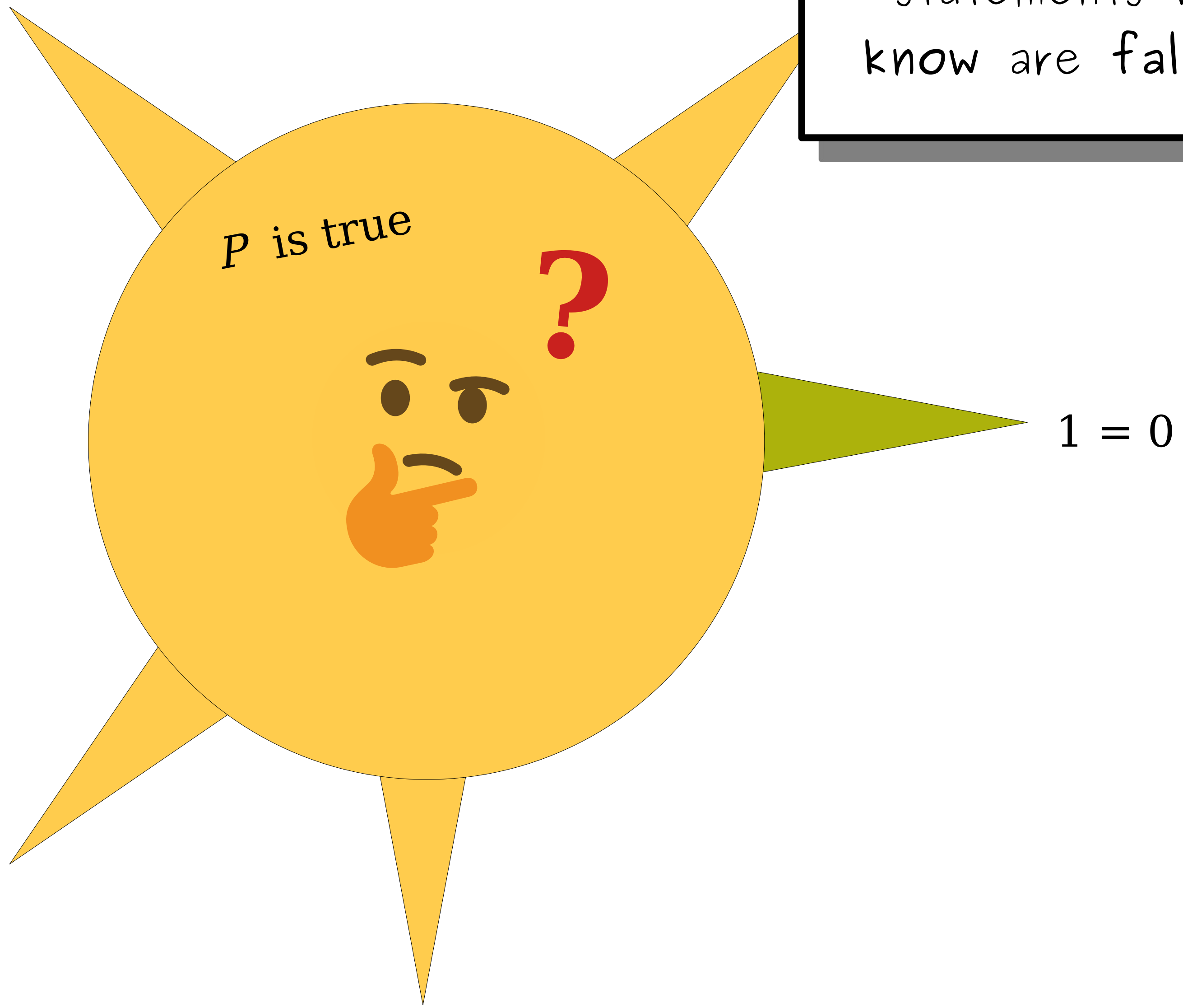

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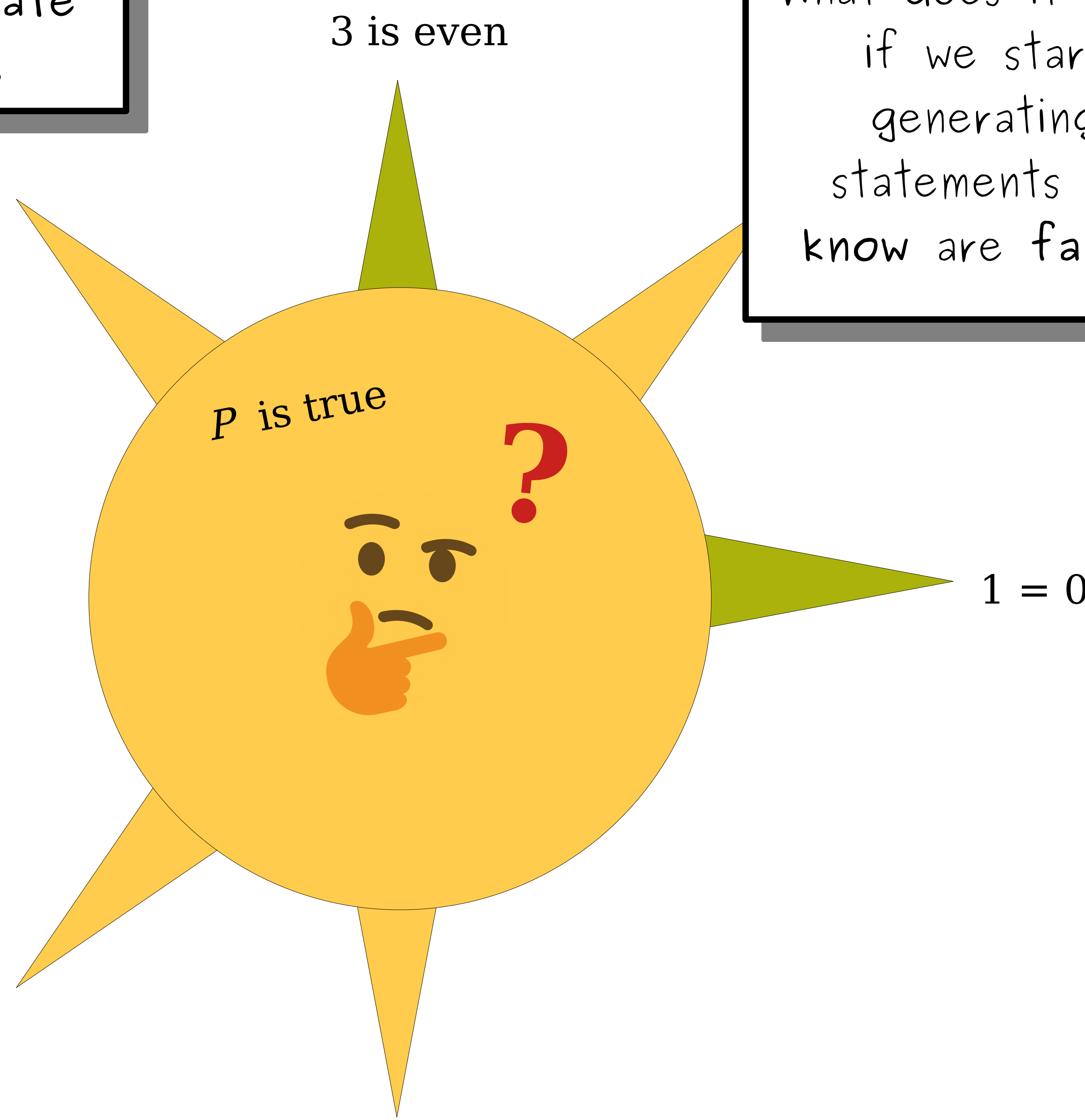

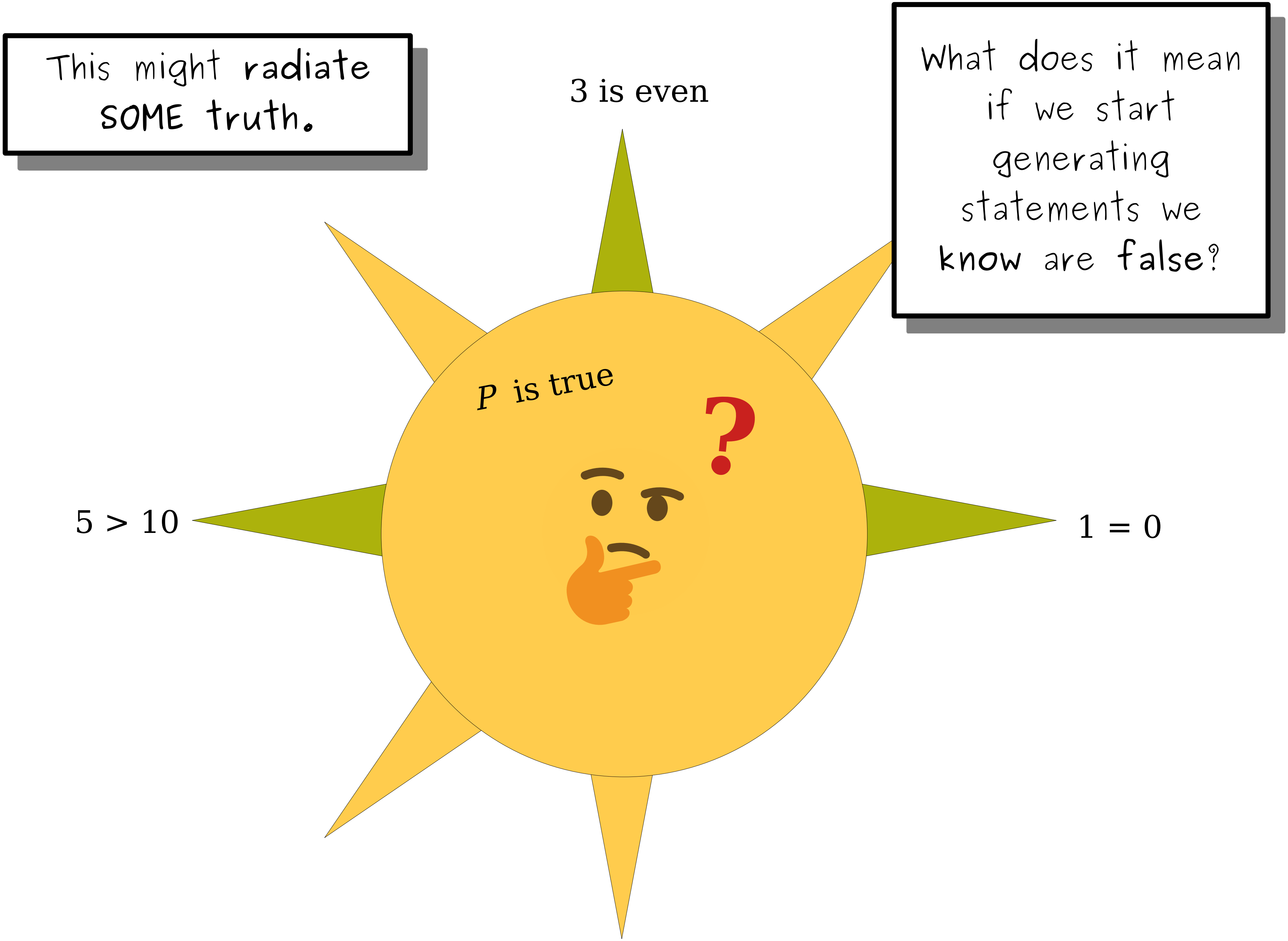
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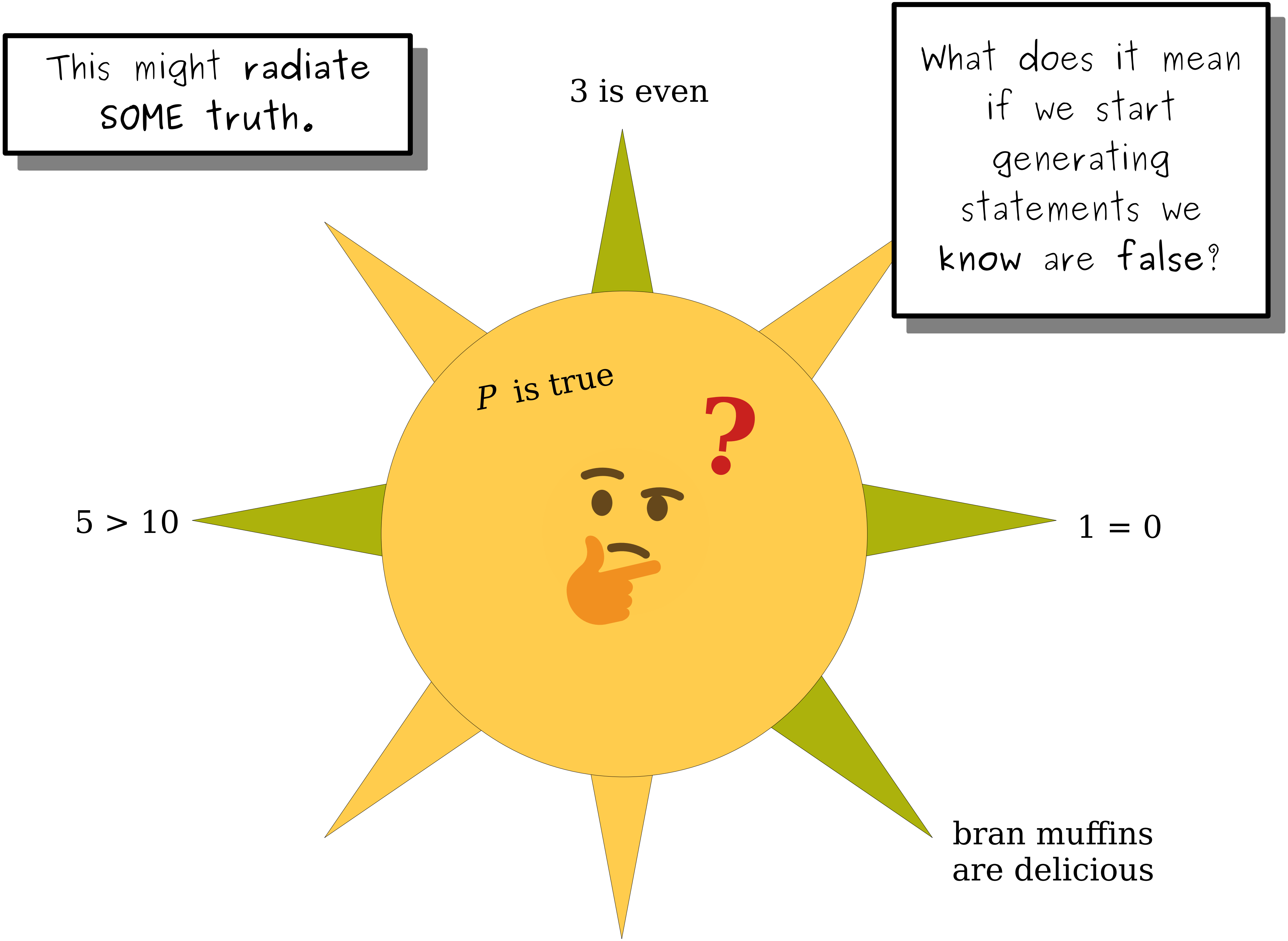
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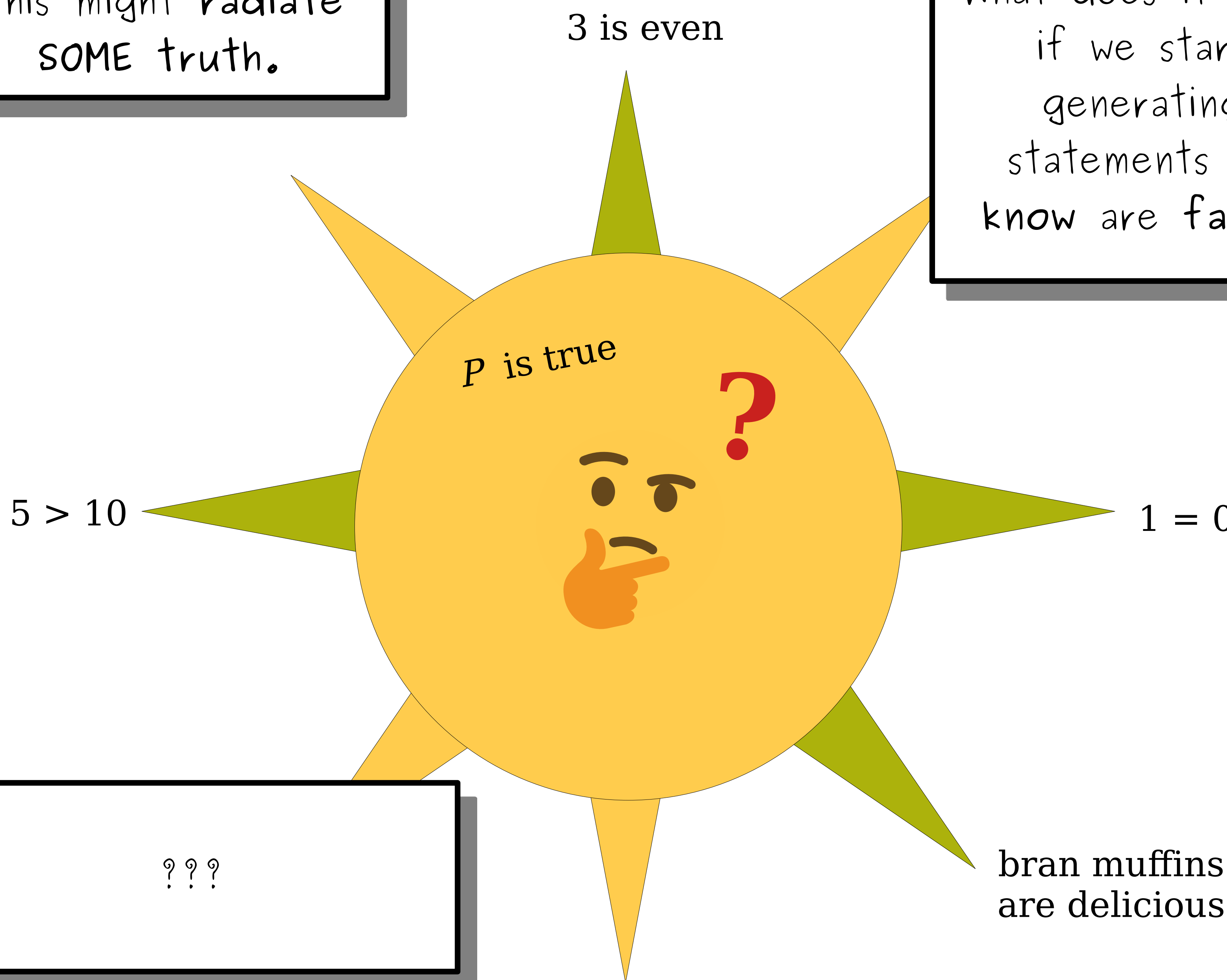

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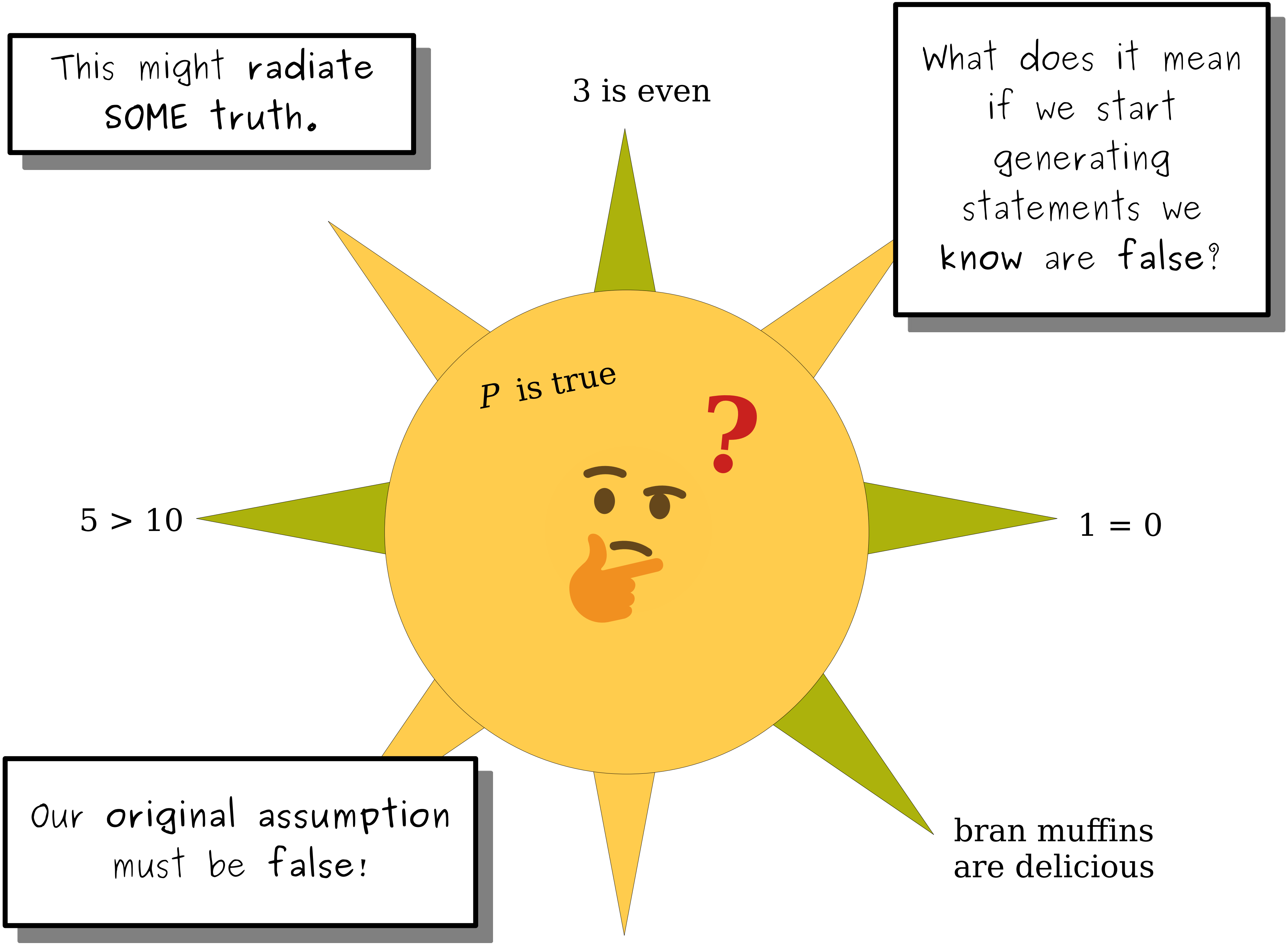
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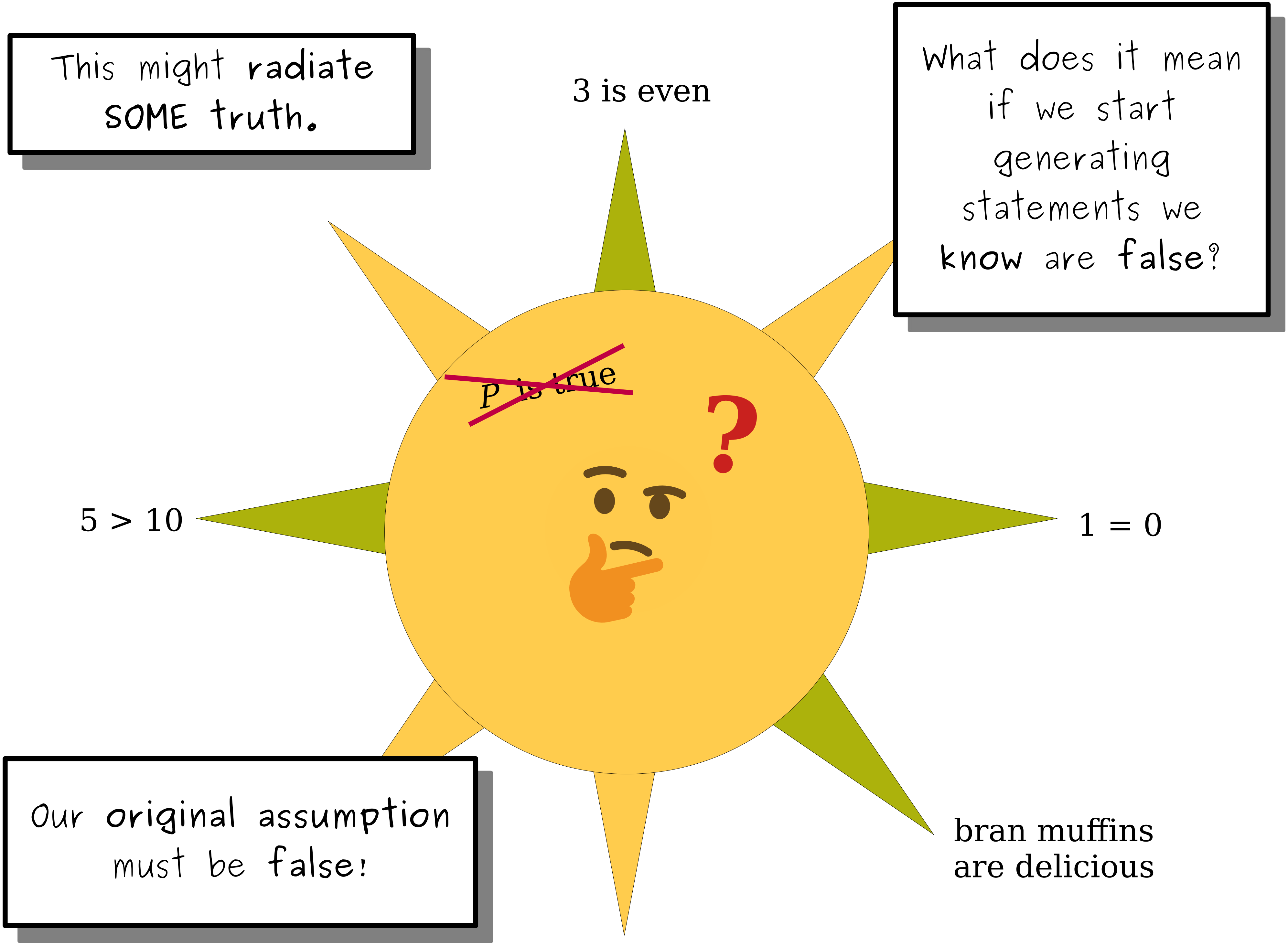
Our original assumption must be false!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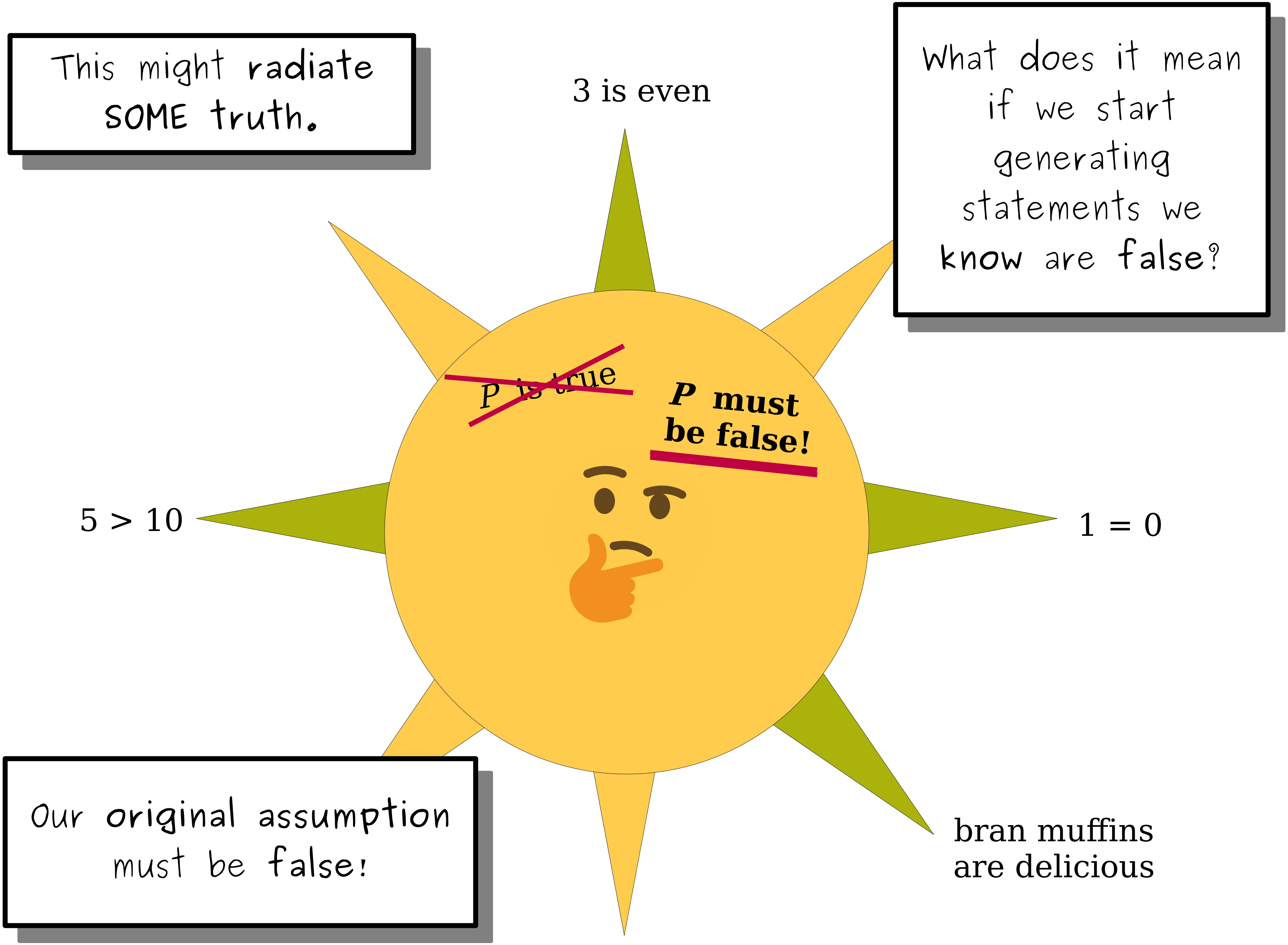
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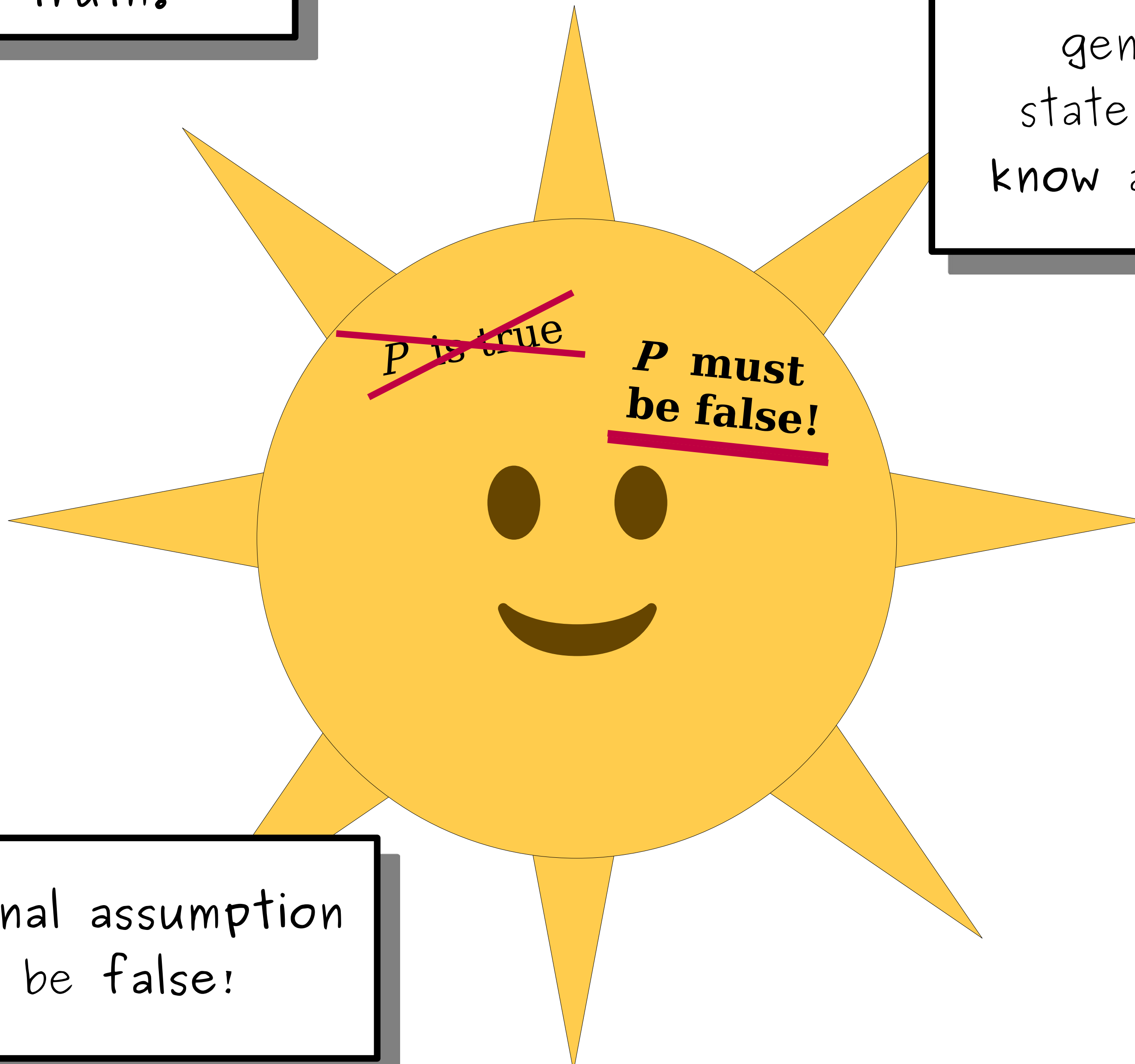

Our **original** assumption must be **false**!

This gives rise to a powerful proof technique called **proof by contradiction**!

Suppose we want to use this technique to show that $\boldsymbol{P}$ is **true**.

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What proposition can we place in the Zone of Uncertainty to accomplish this?

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Answer:

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Answer: The negation of $\boldsymbol{P}$!

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Suppose we want to use this technique to show that $\boldsymbol{P}$ is **true**.

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i.e., $\boldsymbol{P}$ is **false**

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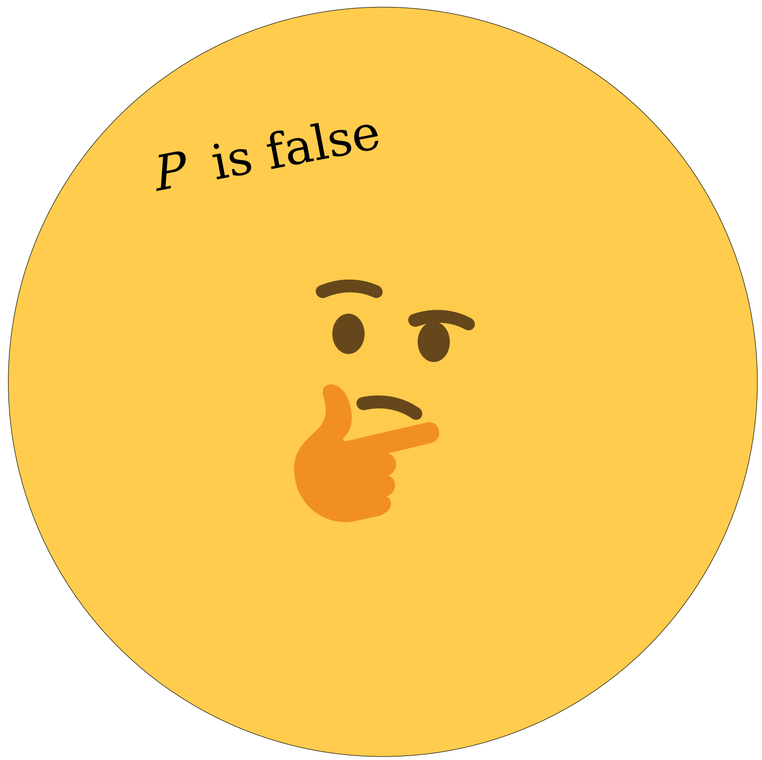
Answer: The negation of ***P*** !

i.e., ***P*** is **false**



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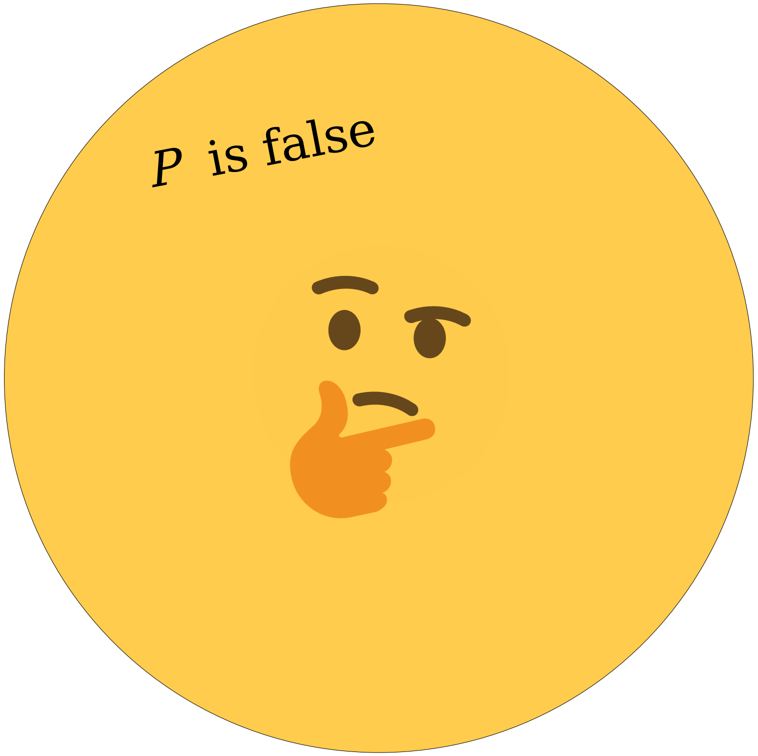

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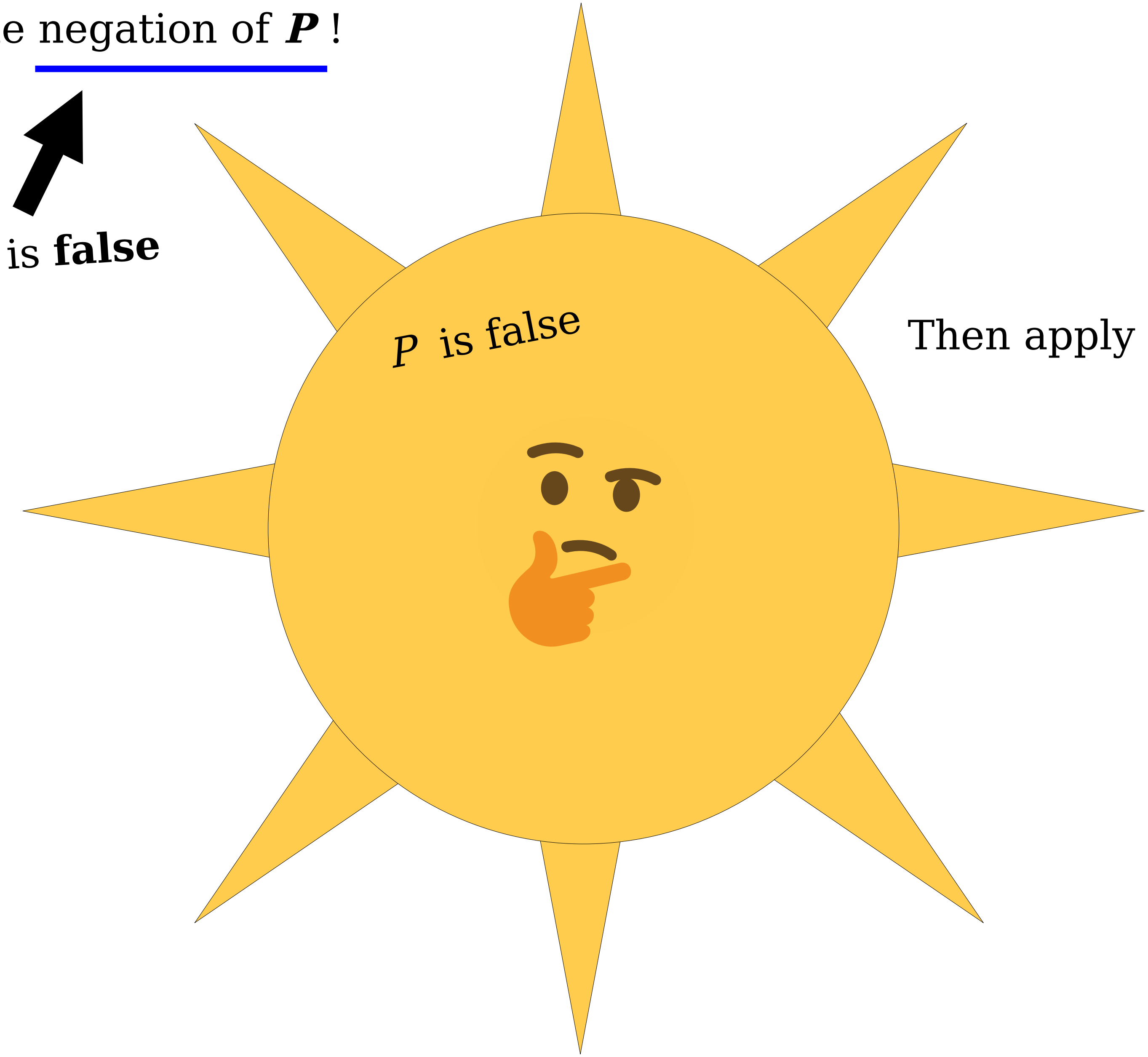

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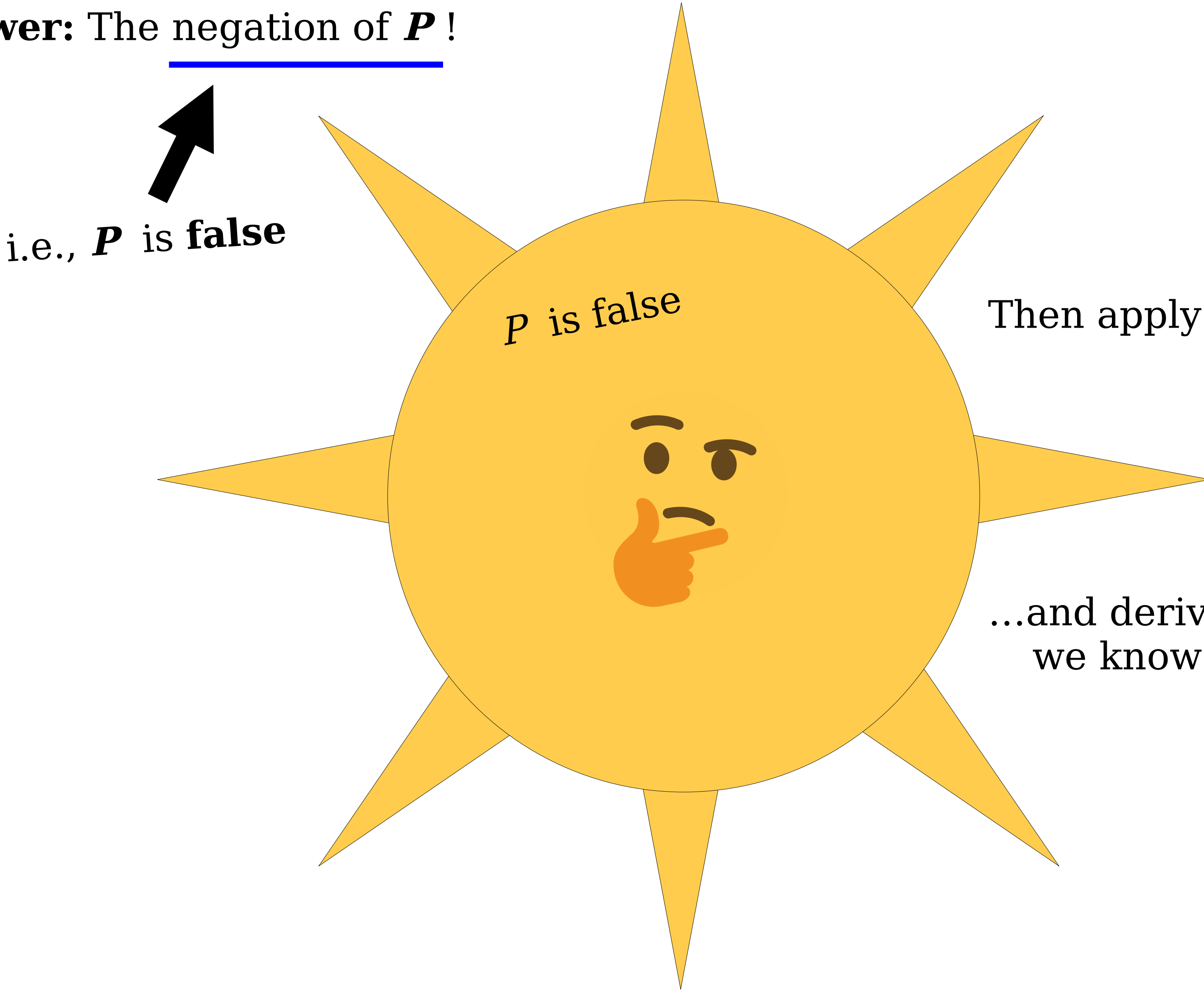

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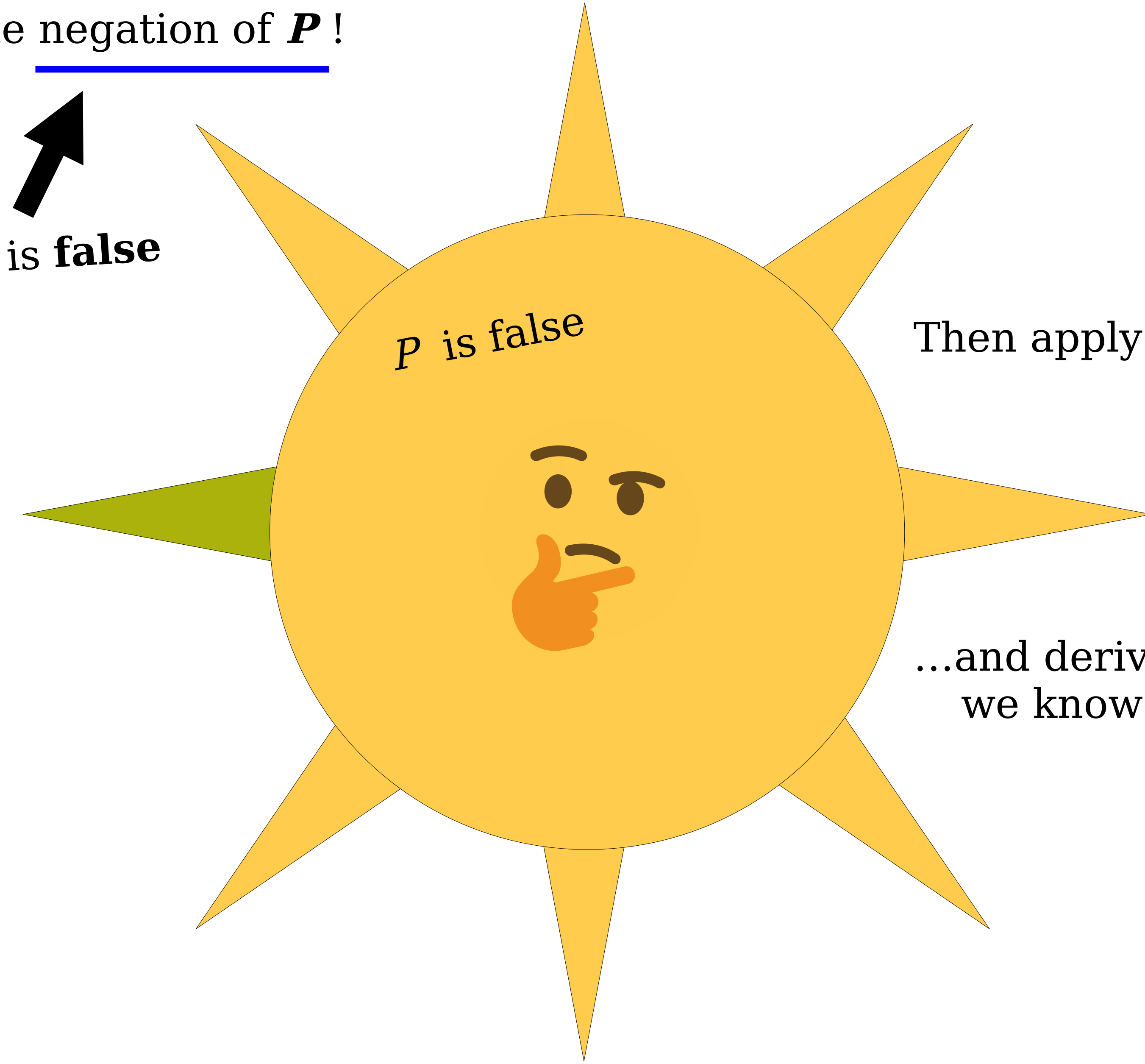

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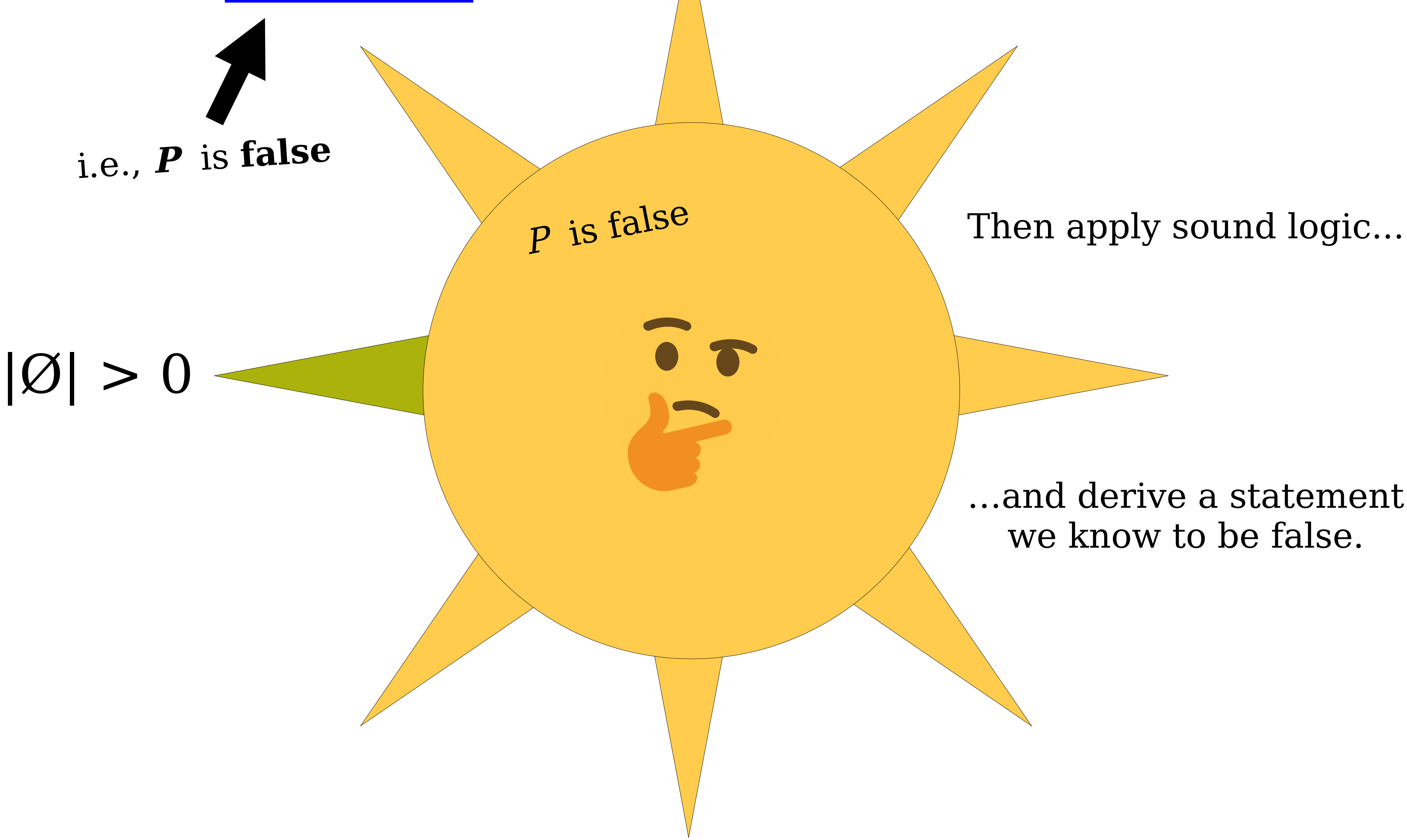

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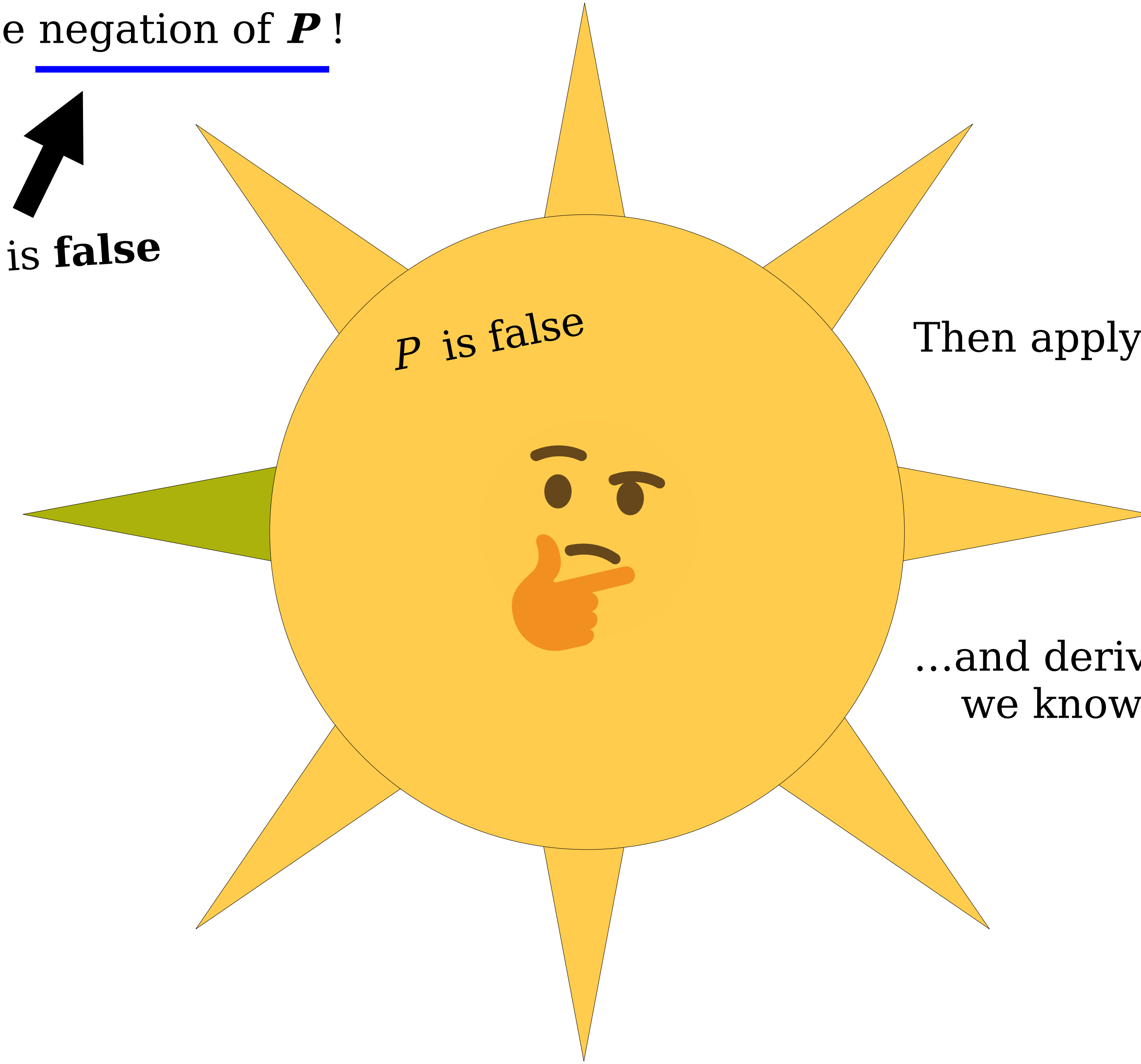

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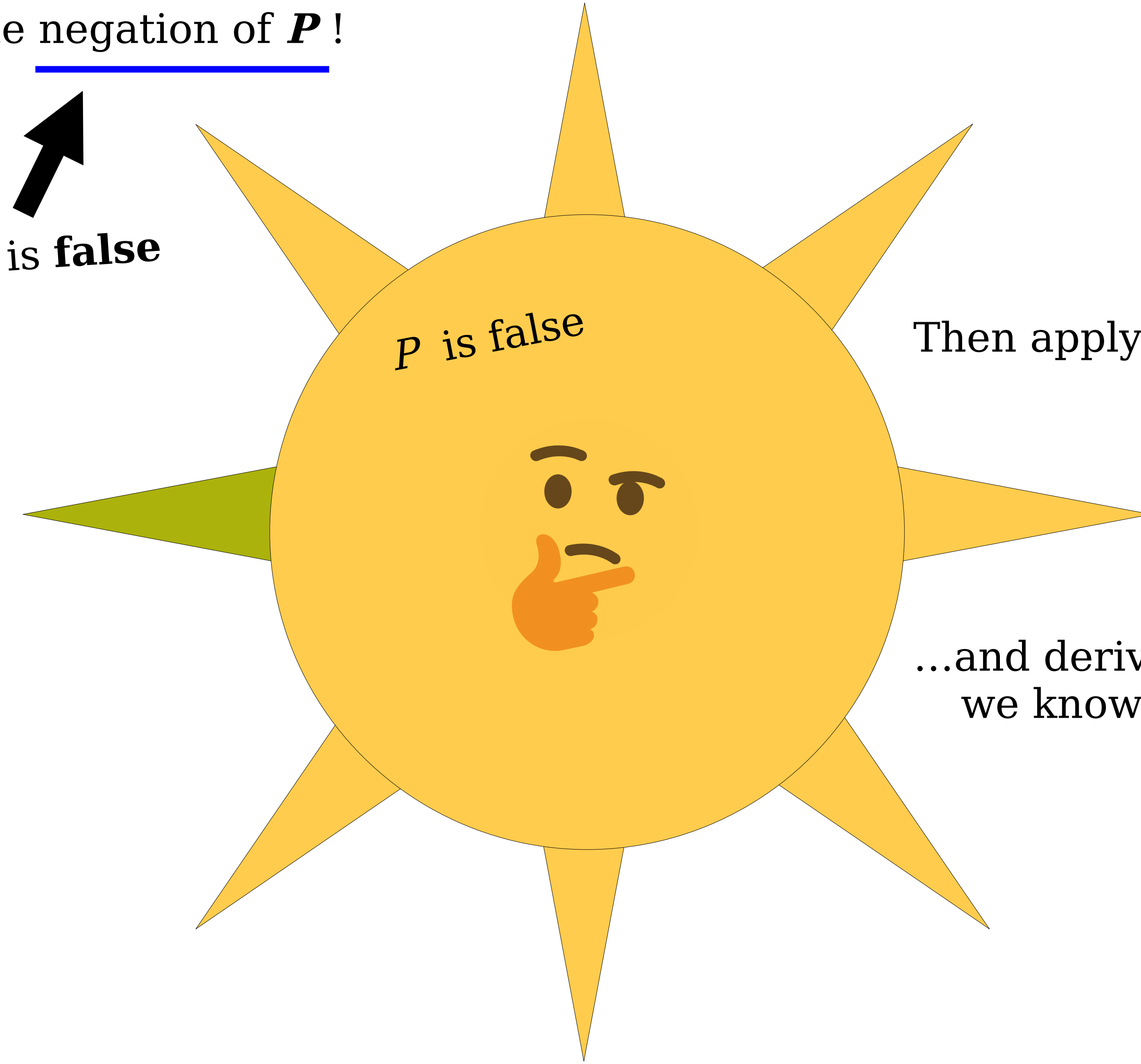

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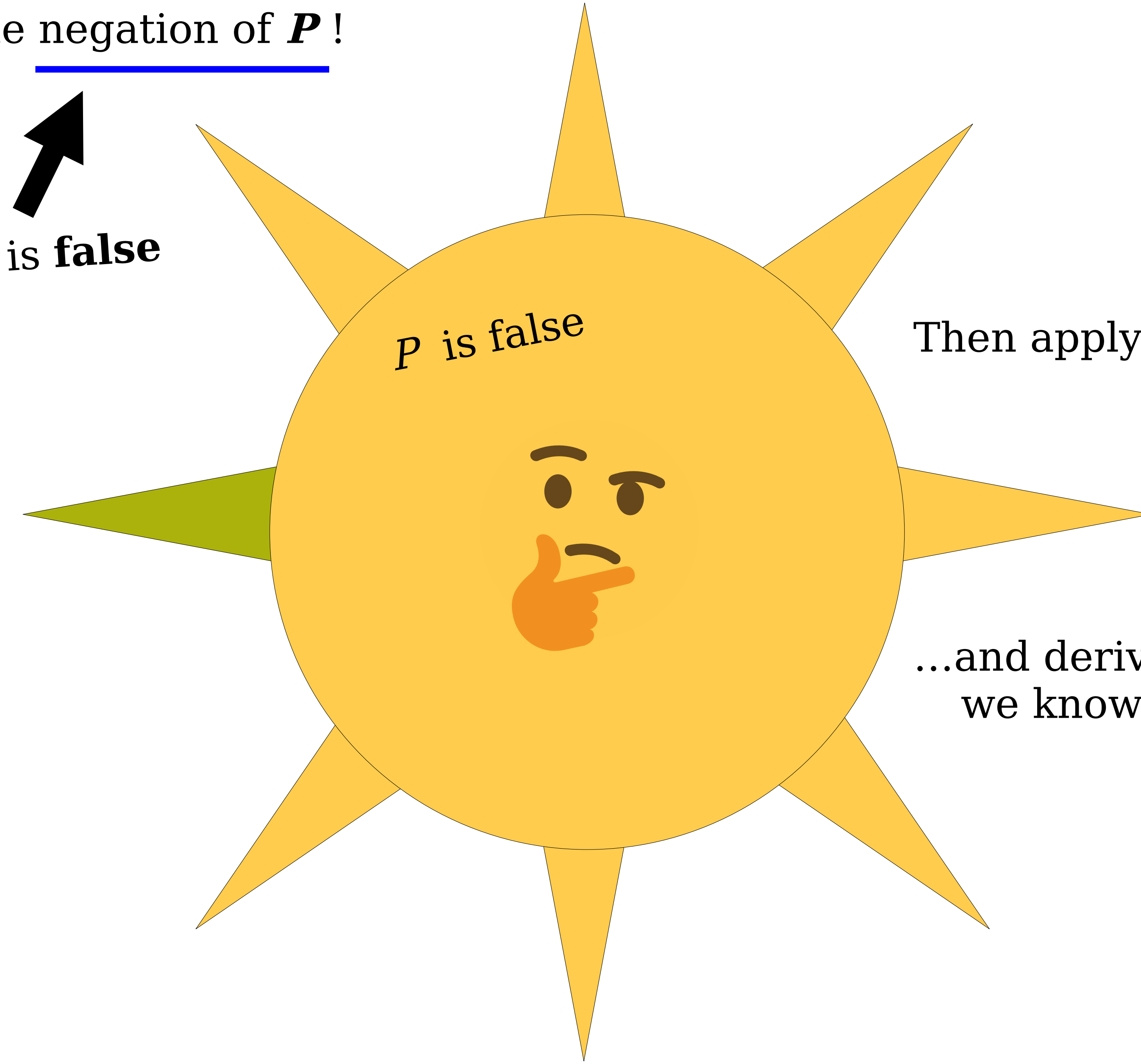

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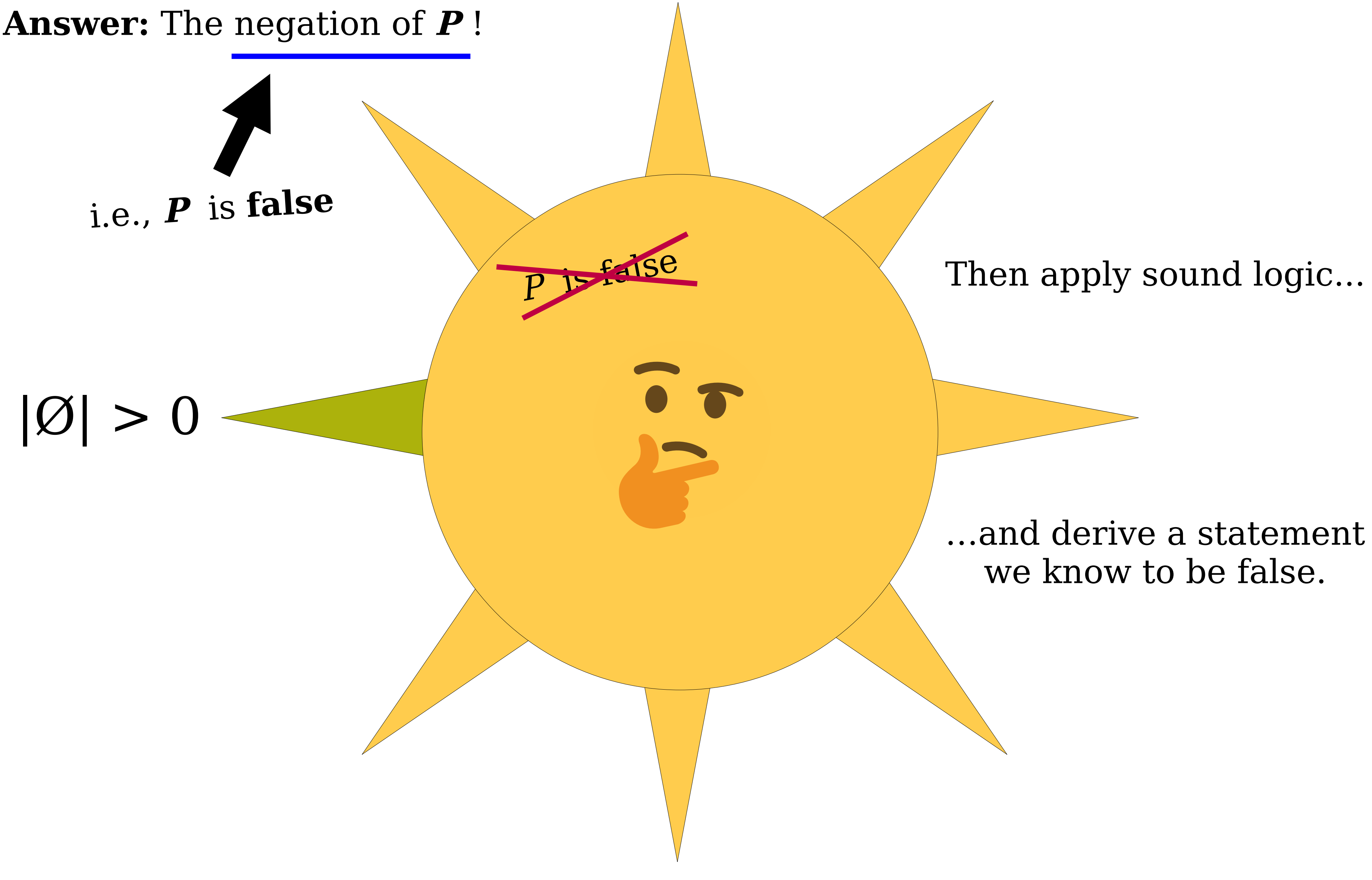

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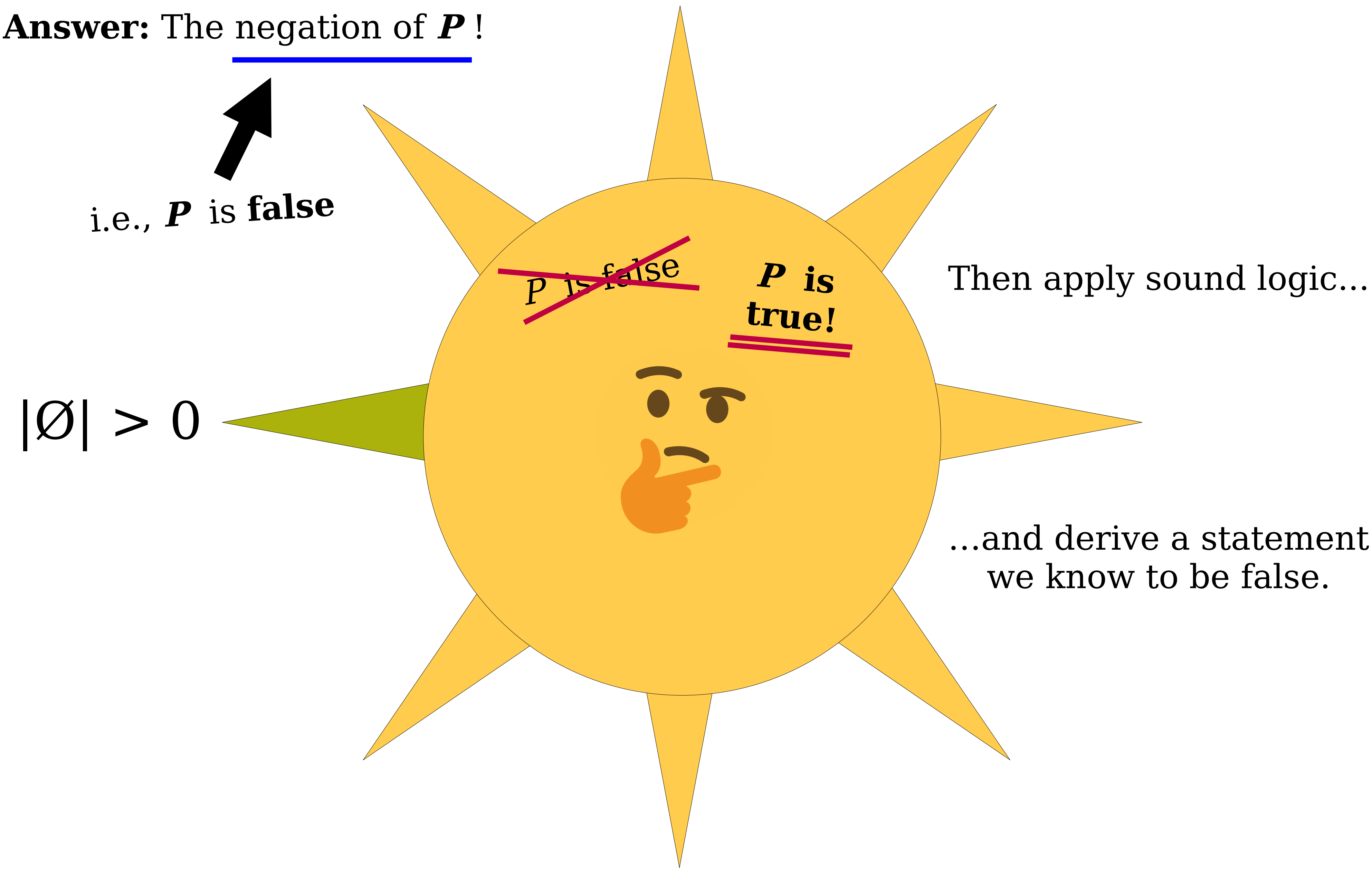

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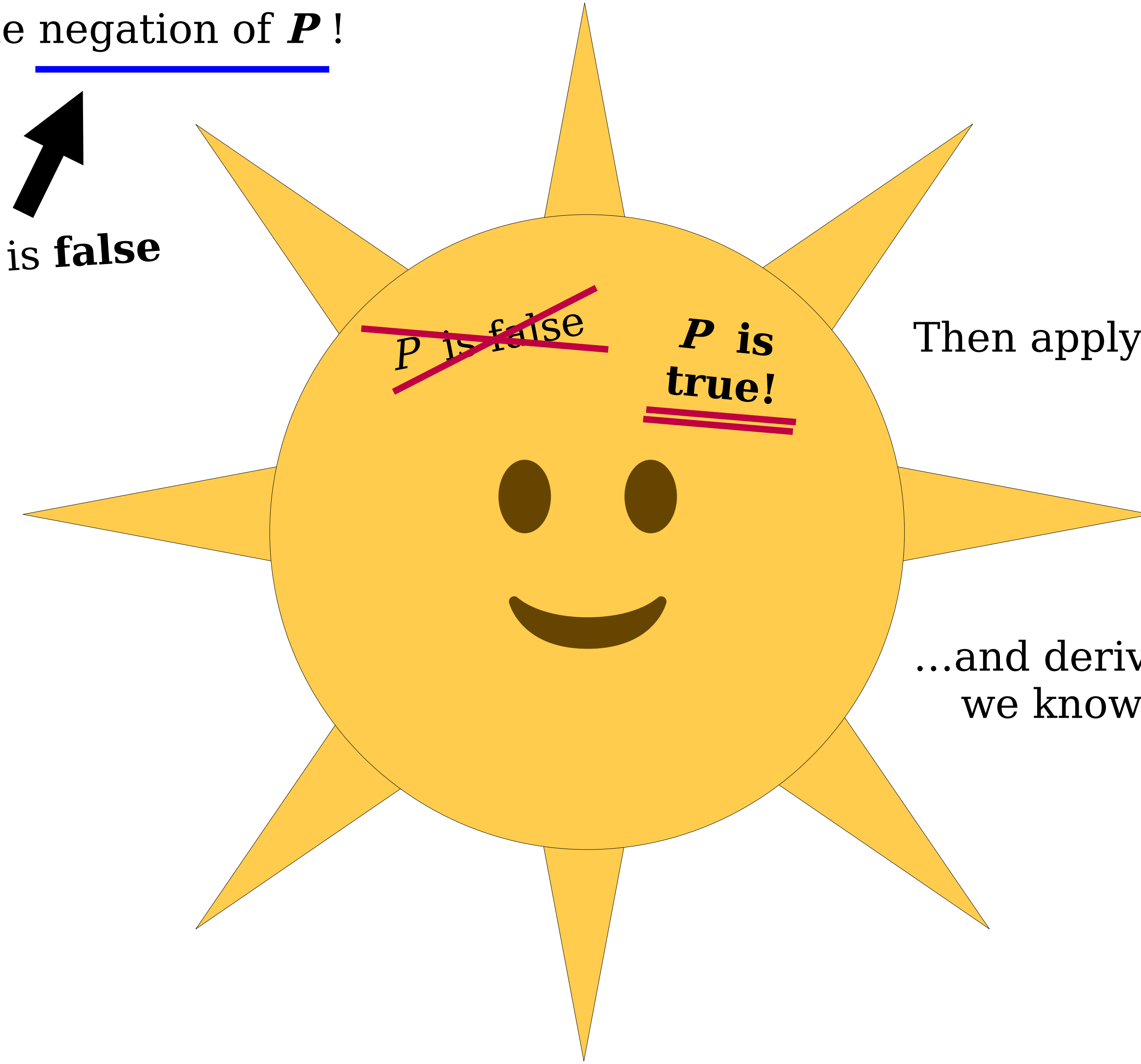

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Summary: Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\varnothing| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
  - $|\wp(\mathbb{N})| > |\mathbb{N}|$
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

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Proof:

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To prove this statement by contradiction, we're going to assume its negation.

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What is the negation of the statement "there is no largest set?"

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What is the negation of the statement "there is no largest set?"

One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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Now, consider the set $\wp(S)$.

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Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S.

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Another Example

Latin Squares

- A ***Latin square*** is an $n \times n$ grid filled with the numbers 1, 2, …, n such that every number appears in each row and each column exactly once.

1	2	3
2	3	1
3	1	2

1	3	4	2
4	2	1	3
2	1	3	4
3	4	2	1

1	3	5	2	4
3	1	4	5	2
4	5	2	3	1
5	2	1	4	3
2	4	3	1	5

3	2	1	4	5	6
2	4	6	1	3	5
5	6	4	3	2	1
4	1	5	2	6	3
6	3	2	5	1	4
1	5	3	6	4	2

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2	1	3	4
3	4	2	1

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3	1	4	5	2
4	5	2	3	1
5	2	1	4	3
2	4	3	1	5

3	2	1	4	5	6
2	4	6	1	3	5
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4	2	5	3	1
3	4	2	1	5

Latin Squares

- Notice anything about what's on the main diagonals of these symmetric Latin squares?
- ***Theorem:*** Every odd-sized symmetric Latin square has every number 1, 2, …, n on its main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

3	2	5	1	4
2	1	4	5	3
5	4	2	3	1
1	5	3	4	2
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Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \ldots, n$ on its main diagonal.

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What is the negation of the theorem?

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One option:

There is a symmetric Latin square of odd size n × n that does not have one of the numbers 1, 2, …, n on its main diagonal.

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(Intermission)

Time-Out for Announcements!

Problem Set One

- Problem Set One goes out today. It's due next Friday at 1:00PM.
  - Explore the language of set theory and better intuit how it works.
  - Learn more about the structure of mathematical proofs.
  - Write your first "freehand" proofs based on your experiences.
- As always, start early, and reach out if you have any questions!

Office Hours

- It is ***completely normal*** in this class to need to get help from time to time.
- Feel free to ask clarifying and conceptual questions on EdStem.
- Need more structured help? We have office hours! Feel free to stop on by.
  - Check out the online “Guide to Office Hours” for more information about how our office hours system works.
  - The OH calendar will soon be available on the course website.
- Office hours start this Sunday.

Vaccines!

- It's Vaccine Season! Yay! What a great way to protect yourself and others.
- You can get a free flu shot through Vaden. Details are at this link:

  https://ehs.stanford.edu/flu/information

- Stanford Health Care offers free bivalent COVID booster vaccines. Use this link to create an account to sign up:

  https://myhealth.stanfordhealthcare.org/

- Santa Clara County (where Stanford is located) also offers flu shots, COVID vaccines, and COVID boosters. Details and appointments here:

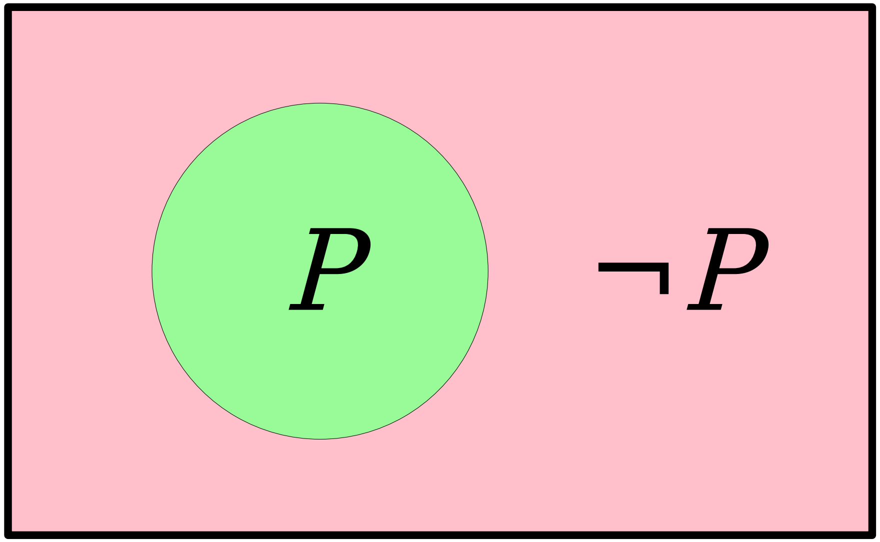
  https://vax.sccgov.org/

Readings for Today

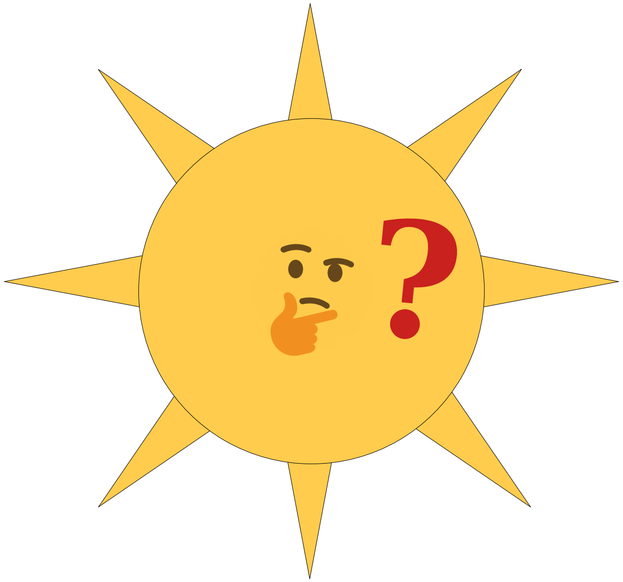
- On the course website we have some information you should look over.
- First is the ***Proofwriting Checklist***. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.
- Next is the ***Guide to Office Hours***, which talks about how our office hours work and how to make the most effective use of them.
- Finally is the ***Guide to LaTeX***, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.

(the lights flash in the atrium)

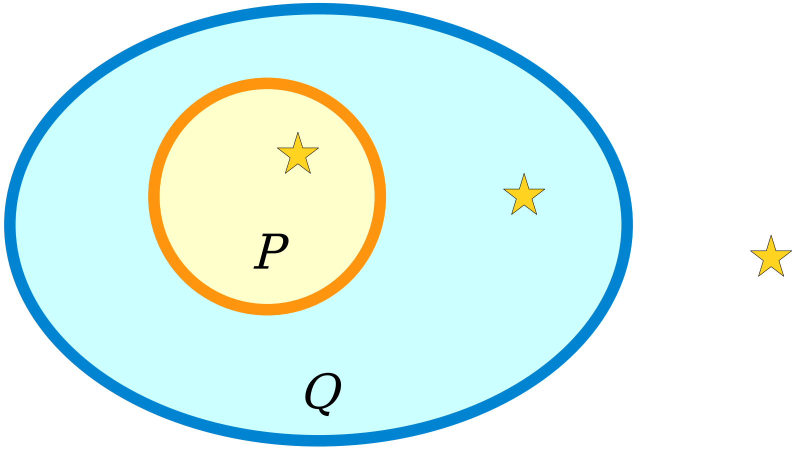
Back to CS103!



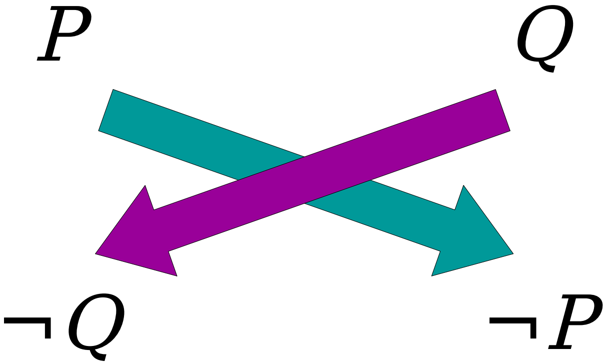
Logical Negation



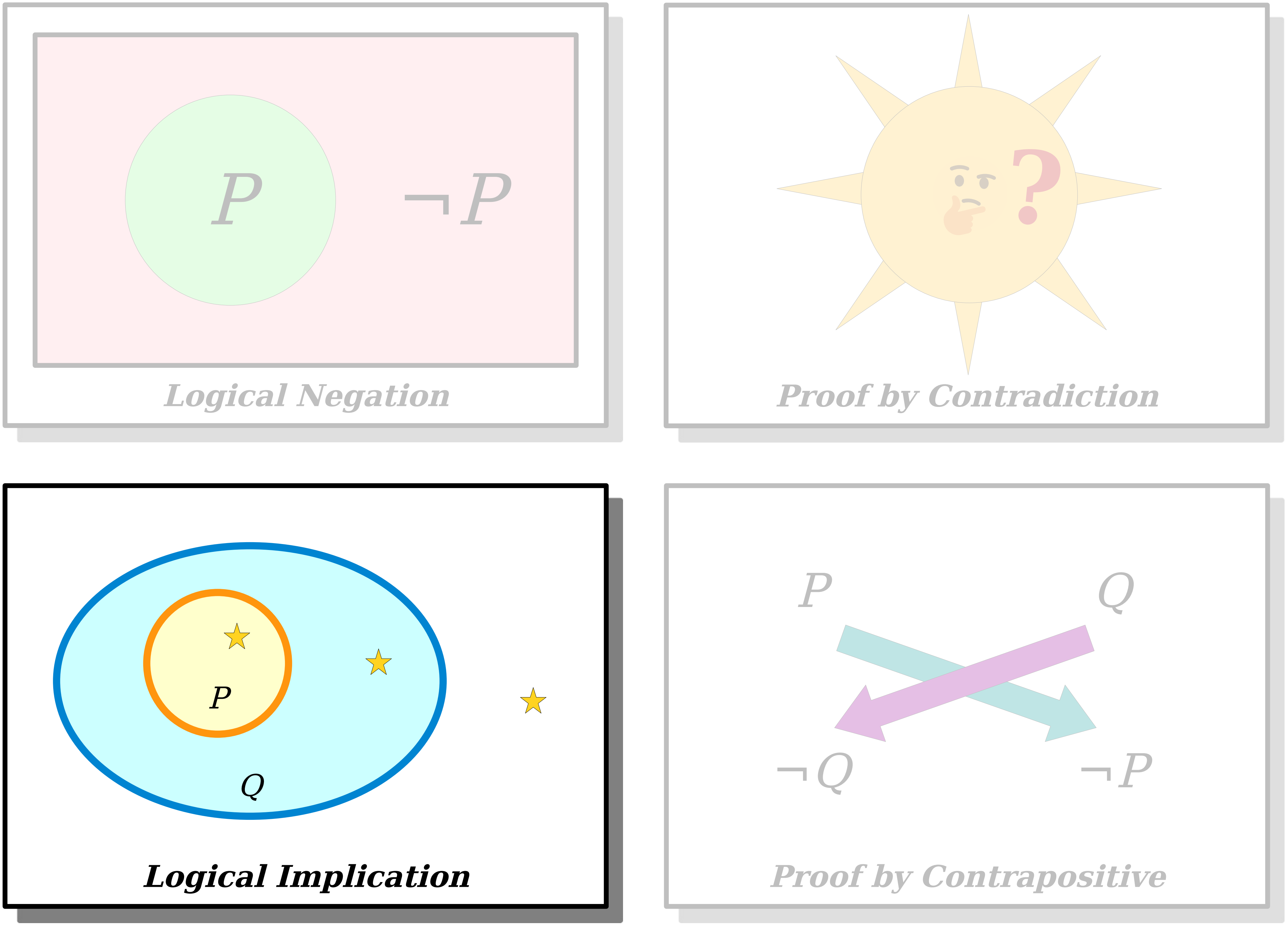
Proof by Contradiction



Logical Implication



Proof by Contrapositive



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Proof by Contradiction
P
Q
Logical Implication
P
Q
¬Q
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Proof by Contrapositive

Act III

Logical Implication

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If *n* is an even integer, then n^2 is an even integer.

This part of the implication is called the *antecedent*.

This part of the implication is called the *consequent*.

An ***implication*** is a statement of the form "If *P* is true, then *Q* is true."

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

Another Example

If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

An ***implication*** is a statement of the form "If P is true, then Q is true."

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If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

Let's explore the definition and nature of implication through this example!

An ***implication*** is a statement of the form "If P is true, then Q is true."

If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

Let's explore the definition and nature of implication through this example!

"If ***P***, then ***Q***."

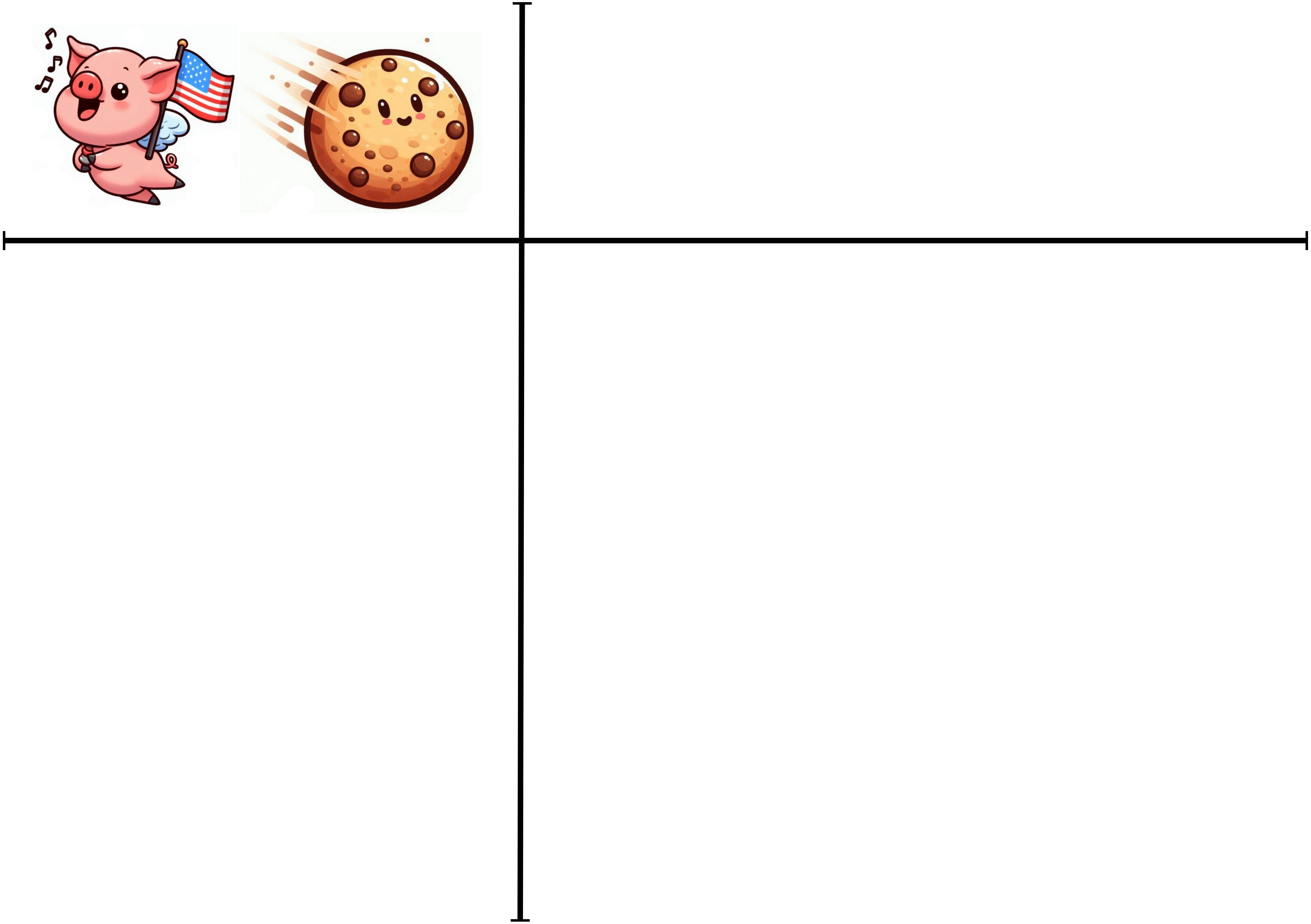
An ***implication*** is a statement of the form "If P is true, then Q is true."

If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

Let's explore the definition and nature of implication through this example!

“If , then .”

An ***implication*** is a statement of the form “If P is true, then Q is true.”

What is the status of our “if , then ” contract?

What is the status of our “if , then ” contract?

What is the status of our “if , then ” contract?

What is the status of our "if , then " contract?

contract is not violated

 	What is the status of our “if , then ” contract?
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	contract **is** violated

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	contract is not violated

What is the status of our
"if , then " contract?

This one often surprises people! It's part of our **definition** of implication and diverges from how conditional statements work in code.

contract is not violated

contract is not violated

	What is the status of our “if , then ” contract?
	contract is not violated
	contract **is** violated
	contract is not violated
	contract is not violated

What is the status of our “if , then ” contract?

This one reveals how to negate an implication!

	contract is not violated
	contract **is** violated
	contract is not violated
	contract is not violated

What is the status of our "if 🐷🇺🇸, then 🍪" contract?

contract is not violated

contract **is** violated

contract is not violated

...olated

This one reveals how to negate an implication!

The only time "if P, then Q" is false is when ***P*** is true and ***Q*** is false.

What Implications Mean

“If there's a rainbow in the sky, then it's raining somewhere.”

- In mathematics, implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
  - Rainbows do not cause rain.

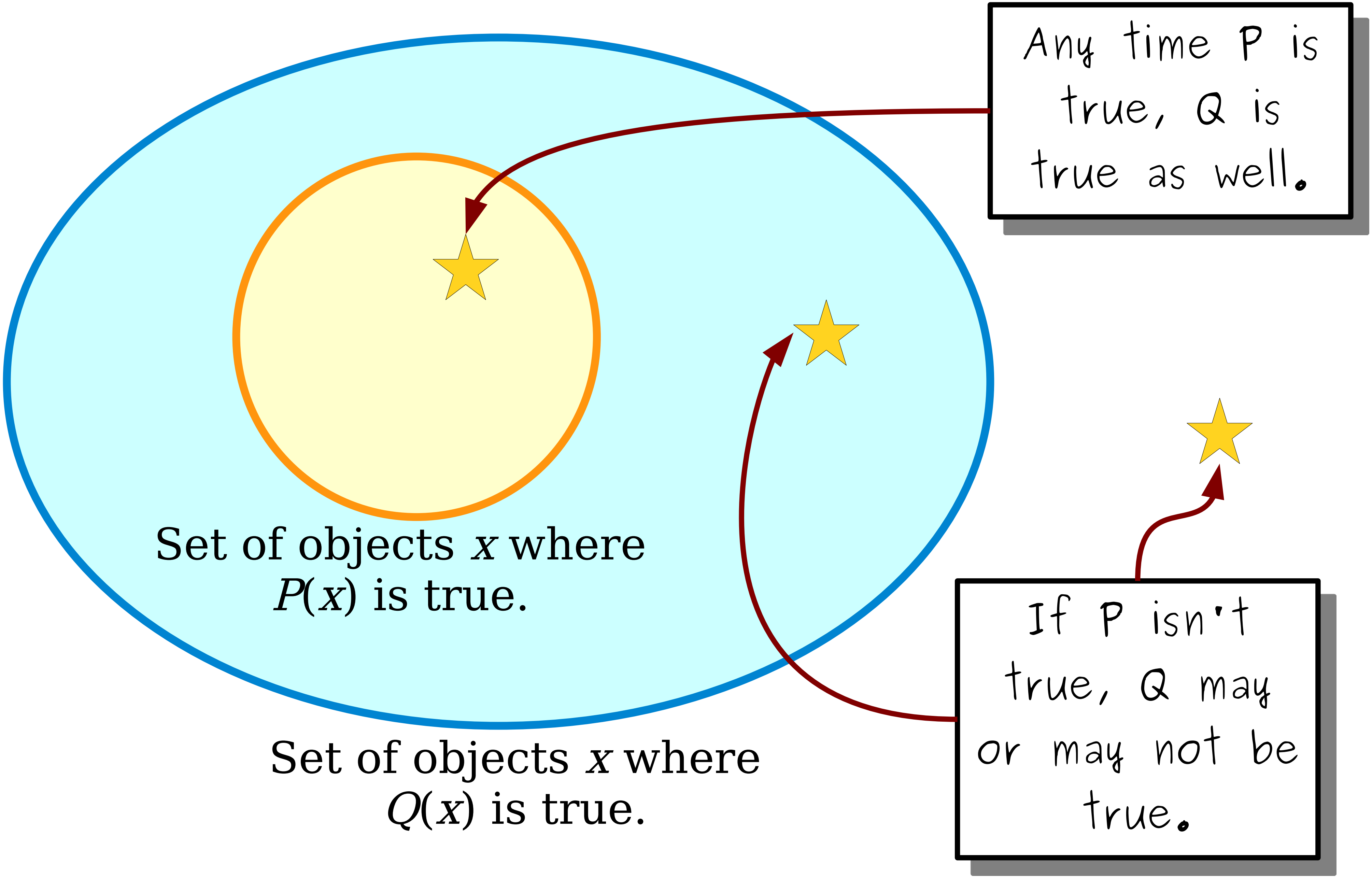
What Implications Mean

- In mathematics, a statement of the form

  For any x, if $P(x)$ is true, then $Q(x)$ is true

  means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



How do you negate an implication?

Consider once again the

“if  , then  ”

contract.

Question: What has to happen for this contract to be broken?

Answer: A flying pig sings the national anthem, but Sean doesn’t throw cookies to the class.

 	What is the status of our "if , then " contract?
	contract is not violated
	contract **is** violated
	contract is not violated
	contract is not violated

	What is the status of our “if , then ” contract?
	contract is not violated
	contract **is** violated
	contract is not violated
	contract is not violated

Key take-away!

The negation of the statement

“For any x, if $P(x)$ is true, then $Q(x)$ is true”

is the statement

“There is at least one x where $P(x)$ is true and $Q(x)$ is false.”

The negation of an implication is not an implication!

Key take-away!

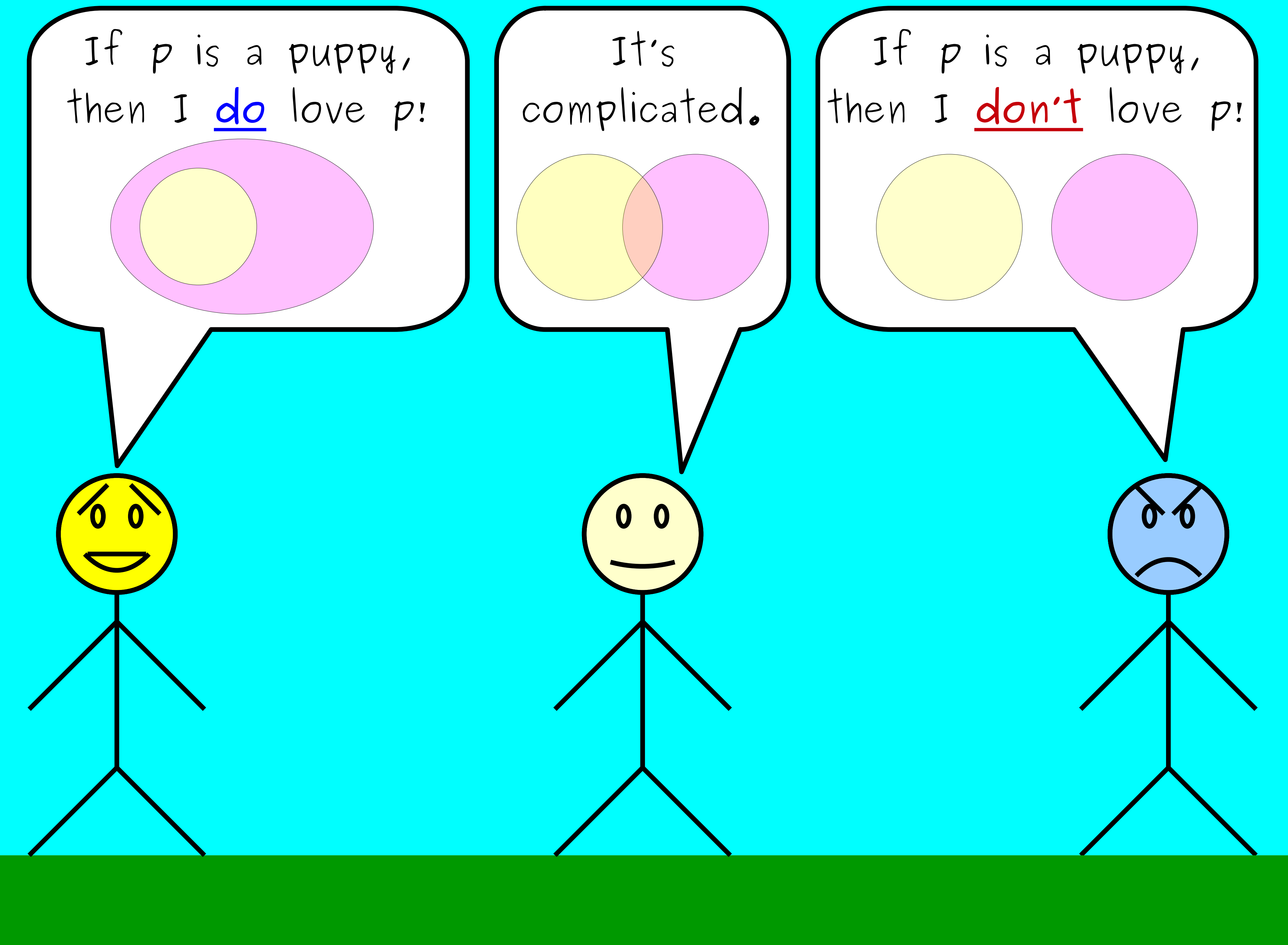
The negation of the statement

"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

is the statement

"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."

The negation of an implication is not an implication!


If p is a puppy, then I do love p!
It's complicated.
If p is a puppy, then I don't love p!

How to Negate Universal Statements:

"For all *x*, *P*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is false."

How to Negate Existential Statements:

"There exists an *x* where *P*(*x*) is true"

becomes

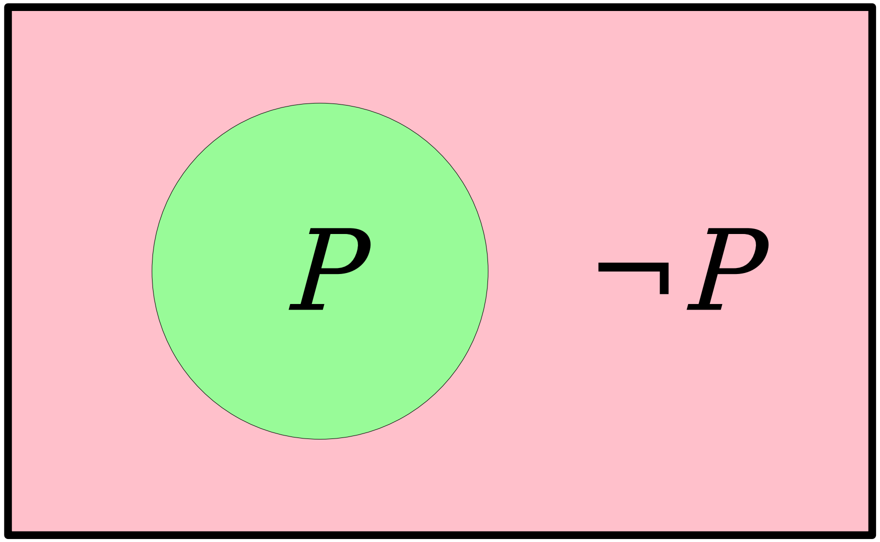
"For all *x*, *P*(*x*) is false."

How to Negate Implications:

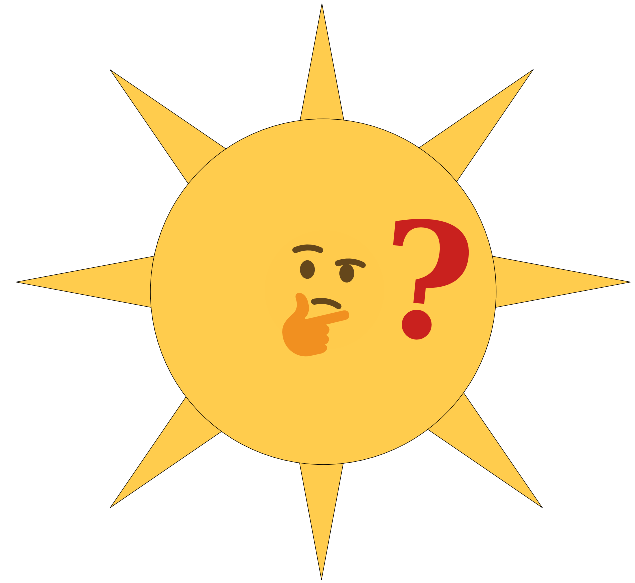
"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

becomes

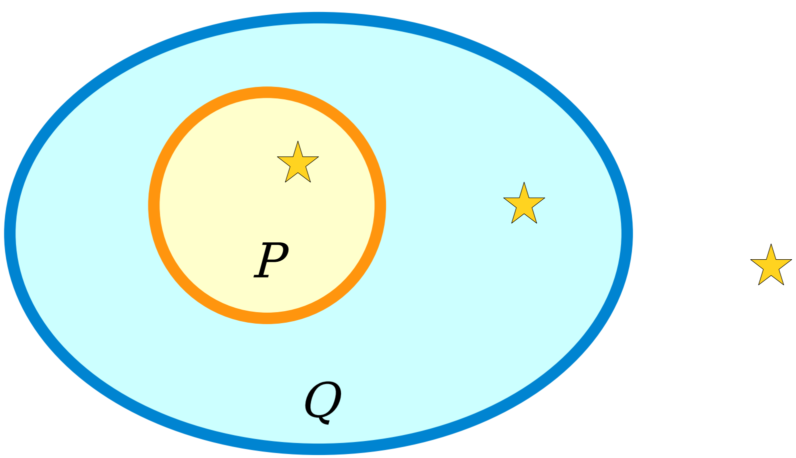
"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false."



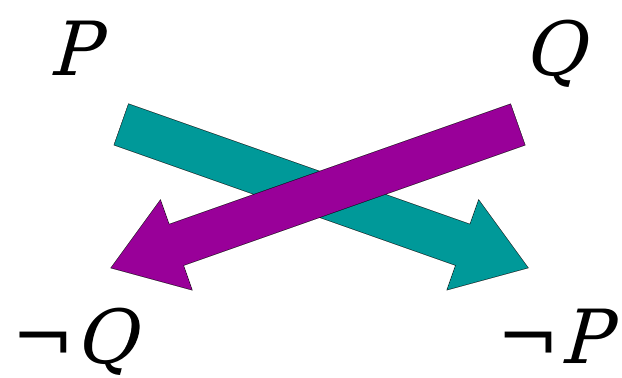
Logical Negation



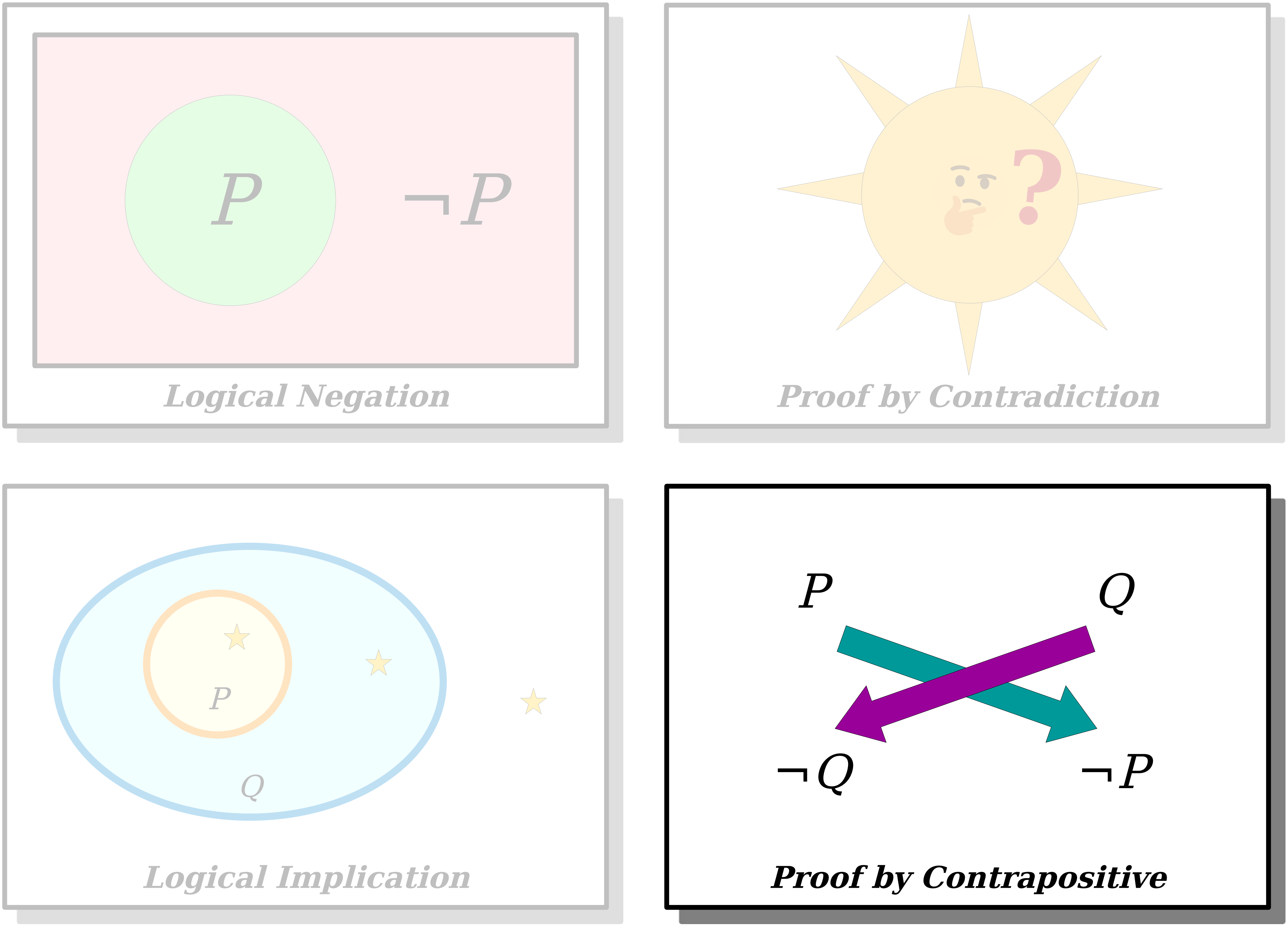
Proof by Contradiction



Logical Implication



Proof by Contrapositive



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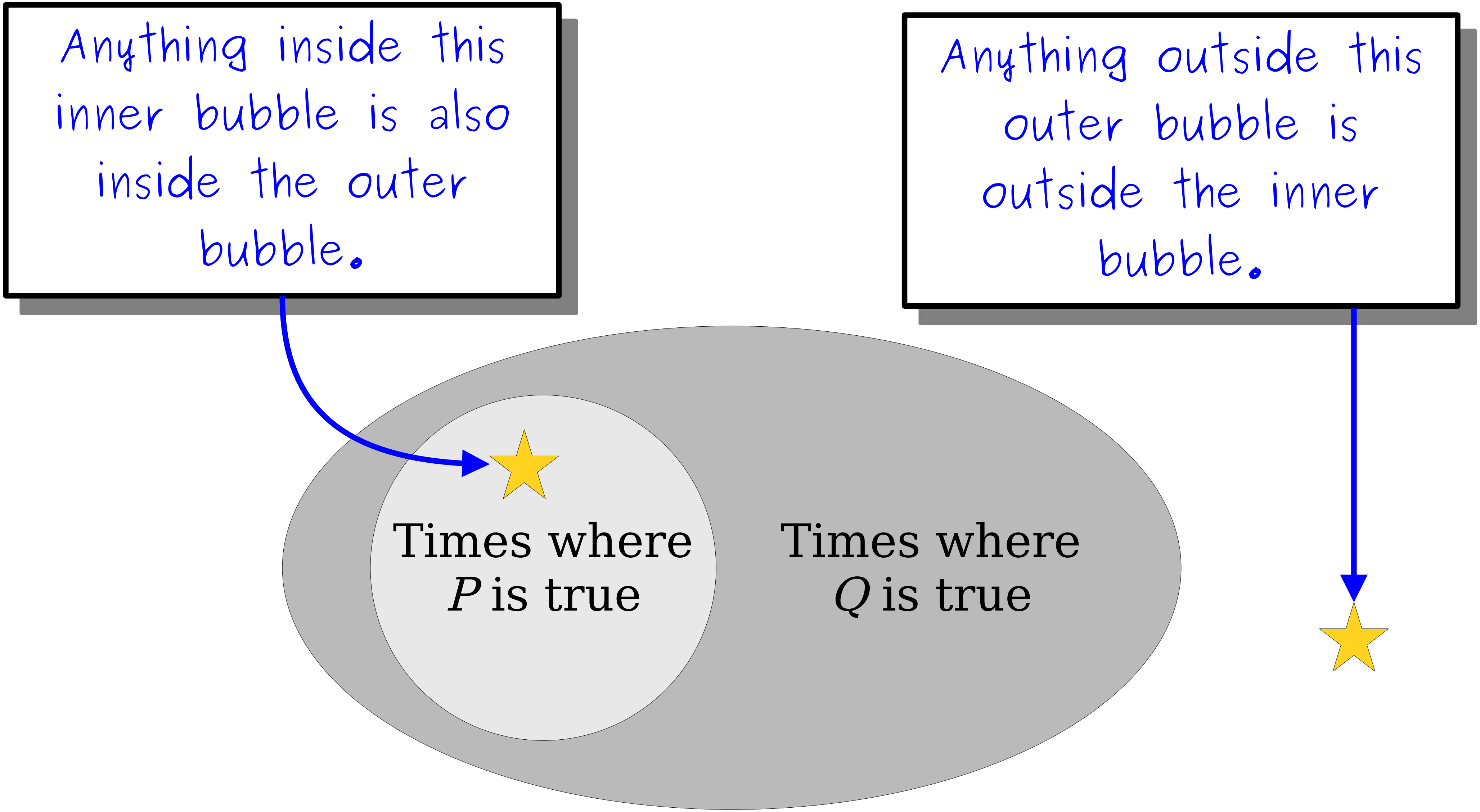
Proof by Contrapositive

Act IV

Proof by Contrapositive



If P is true, then Q is true.

If Q is false, then P is false.

The Contrapositive

- The ***contrapositive*** of the implication

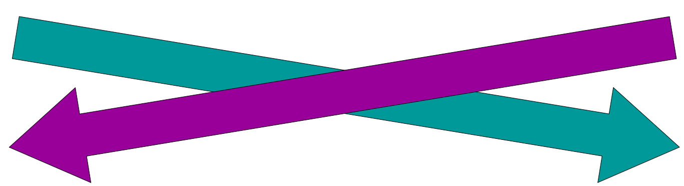
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

The Contrapositive

- The ***contrapositive*** of the implication

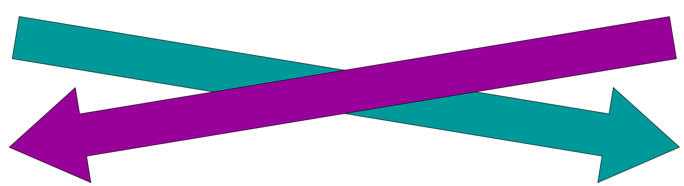
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If I store cat food inside, then raccoons won’t steal it.



If raccoons stole the cat food, then I didn’t store it inside.

To prove the statement

"if *P* is true, then *Q* is true,"

you can choose to instead prove the equivalent statement

"if *Q* is false, then *P* is false,"

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: We will prove the contrapositive of this statement.

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Proof: We will prove the contrapositive of this statement.

This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

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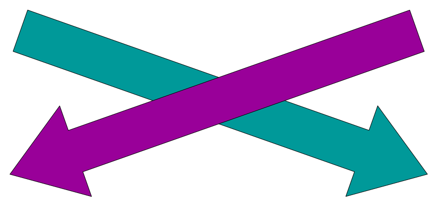

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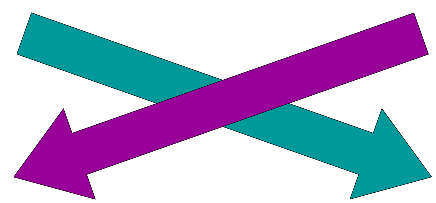


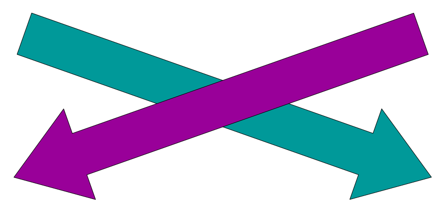
If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.



If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$n^2 = (2k + 1)^2$$

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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to show. ■

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

For any integer *n*, if *n* is even, then n^2 is even.

For any integer *n*, if n^2 is even, then *n* is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say

for any integer *n*: *n* is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form

  ***P* if and only if *Q*,**

  you need to prove two separate statements.
  - First, that if P is true, then Q is true.
  - Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

What We Learned

- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What's a proof by contradiction?***
  - It's a proof of a statement P that works by showing that P cannot be false.
- ***What's an implication?***
  - It's statement of the form "if P, then Q," and states that if P is true, then Q is true.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if P, then Q" is "if not Q, then not P.")

Your Action Items

- ***Read "Guide to Office Hours," the "Proofwriting Checklist," and the "Guide to LaTeX."***
  - There's a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we'll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what's being asked. That'll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.

Appendix: Proving Implications by Contradiction

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***
  - ***Proof by Contrapositive.***
  - ***Proof by Contradiction.***

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***
  - ***Proof by Contradiction.***

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P* is true**, then prove ***Q* is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q* is false**, then prove that ***P* is false**.
  - ***Proof by Contradiction.***

    … what does this look like?

Theorem: For any integer n, if n^2 is even, then n is even.

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What is the negation of our theorem?

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what the negation of the original statement is.
3. Say you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

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Proving Implications

- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

- We have three options available to us:
  - ***Direct Proof:***

    Assume ***P*** **is true**, then prove ***Q*** **is true**.
  - ***Proof by Contrapositive.***

    Assume ***Q*** **is false**, then prove that ***P*** **is false**.
  - ***Proof by Contradiction.***

    Assume ***P is true*** and ***Q is false***, then derive a contradiction.